Math200b, lecture 10

Golsefidy

Forgetful and representable functors.

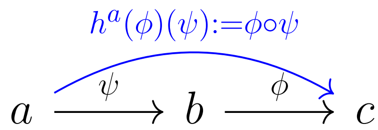
In the previous lecture we defined category and functor. Here are two important functors:

Forgetful functor. Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories such that $\mathrm{Ob}(\mathcal{C}) \subseteq \mathrm{Ob}(\mathcal{D})$ and for any $a, b \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(a, b) \subseteq \mathrm{Hom}_{\mathcal{D}}(a, b)$. Then we can consider $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{F}(a) := a$ and $\mathcal{F}(\phi) := \phi$ for any $\phi \in \mathrm{Hom}_{\mathcal{C}}(a, b)$. If $\mathcal{F}$ is a functor, we call it the forgetful functor. For instance, we have forgetful functors

$$(A\mathbf{-mod}) \to \mathbf{Ab} \to \mathbf{Grp} \to \mathbf{Set};$$

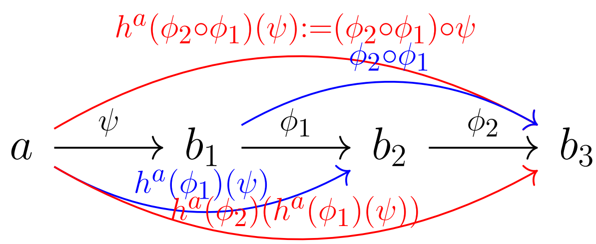
at each level we are forgetting certain extra structures of the objects. Let me illustrate how we have been using the forgetful functor: an isomorphism in category $\mathcal{C}$ means $\phi \in \operatorname{Hom}_{\mathcal{C}}(a, b)$ such that, for some $\psi \in \operatorname{Hom}_{\mathcal{C}}(b, a)$, $\phi \circ \psi = 1_b$ and $\psi \circ \phi = 1_a$. The algebraic categories that we have been working with have a forgetful functor to **Set**; and in all the cases (for groups, rings, and A-modules), based on the first isomorphism theorem, we proved that a homomorphism which is a bijection (this implies an isomorphism in the category of sets) is an isomorphism. This is a common theme: *how much do we actually lose by forgetting parts of our structure?*

Representable functor. One recurrent theme in our classes has been the importance of actions of objects: one can understand an object better by letting it *act*. Both in group theory and ring theory we saw the connection between *actions* of an object a with certain $\operatorname{Hom}_{\mathcal{C}}(a, \bullet)$. We can follow the same idea and for $a \in \operatorname{Ob}(\mathcal{C})$ consider $h^a(b) := \operatorname{Hom}_{\mathcal{C}}(a, b)$. When $\mathcal{C}$ is a locally small category, we get a map from $\operatorname{Ob}(\mathcal{C})$ to $\operatorname{Ob}(\mathbf{Set})$. Next we extend this to a functor; to do so for any $\phi \in \operatorname{Hom}_{\mathcal{C}}(b, c)$, we need to define a function $h^a(\phi) : h^a(b) \to h^a(c)$. The next diagram is very suggestive of the following definition $h^a(\phi)(\psi) := \phi \circ \psi$.



Next we check that $h^a : \mathcal{C} \to \mathbf{Set}$ is a functor and it is called a representable functor; suppose $\phi_1 \in \mathrm{Hom}_{\mathcal{C}}(b_1, b_2)$ and $\phi_2 \in \mathrm{Hom}_{\mathcal{C}}(b_2, b_3)$; then

$$\begin{aligned} h^a(\phi_2 \circ \phi_1)(\psi) &= (\phi_2 \circ \phi_1) \circ \psi = \phi_2 \circ (\phi_1 \circ \psi) \\ &= h^a(\phi_2)(h^a(\phi_1)(\psi)) = (h^a(\phi_2) \circ h^a(\phi_1))(\psi); \end{aligned}$$



One can also see that $h^a(1_b) = \mathrm{id}_{h^a(b)}$.

Natural transformation.

Before we go back to module theory, let us define another important concept from category theory: natural transformation. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are two functors from $\mathcal{C}$ to $\mathcal{D}$; then

$\eta : \mathcal{F}_1 \to \mathcal{F}_2$ is called a natural transformation if, for any $a \in \mathrm{Ob}(\mathcal{C})$, $\eta_a \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1(a), \mathcal{F}_2(a))$ and the following diagrams are commutative for any $\phi \in \mathrm{Hom}_{\mathcal{C}}(a, b)$:

$$\begin{array}{ccc} \mathcal{F}_1(a) & \xrightarrow{\mathcal{F}_1(\phi)} & \mathcal{F}_1(b) \\ \downarrow \eta_a & & \downarrow \eta_b \\ \mathcal{F}_2(a) & \xrightarrow{\mathcal{F}_2(\phi)} & \mathcal{F}_2(b) \end{array}$$

(Notice that $\eta_a : \mathcal{F}_1(a) \to \mathcal{F}_2(a)$ kind of justifies the notation $\eta : \mathcal{F}_1 \to \mathcal{F}_2$.) When η_a's are isomorphisms, we say $\mathcal{F}_1(a)$ is naturally isomorphic to $\mathcal{F}_2(a)$.

Representable functors of A-mod.

For a left A-module M, we know that, for any left A-module N, $h^M(N) := \mathrm{Hom}_A(M, N)$ is an abelian group. Next we show that h^M can be promoted to a functor to category of abelian groups.

Lemma 1 *For a (left) A-module M, $h^M : A$-**mod** $\to$ **Ab** is a functor.*

Proof. Since we already know that $h^M : A$-**mod** $\to$ **Set** is a functor and $h^M(N)$ is an abelian group, it is enough to show that $h^M(\phi)$ is an abelian group homomorphism for any $\phi \in \mathrm{Hom}_A(N, N')$:

$$\begin{aligned} h^M(\phi)(\psi_1 + \psi_2) =& \phi \circ (\psi_1 + \psi_2) = \phi \circ \psi_1 + \phi \circ \psi_2 \\ =& h^M(\phi)(\psi_1) + h^M(\phi)(\psi_2). \end{aligned}$$

■

Next we investigate whether injective or surjective maps are sent to injective or surjective maps, respectively.

Lemma 2 *Suppose M, N, N' are (left) A-modules. If $0 \to N \xrightarrow{\phi} N'$ is an exact sequence, then $0 \to h^M(N) \xrightarrow{h^M(\phi)} h^M(N')$ is an exact sequence.*

Proof. Suppose $h^M(\phi)(\psi) = 0$; then for any $x \in M$, $(h^M(\phi)(\psi))(x) = 0$ which implies $\phi(\psi(x)) = 0$. Since ϕ is injective, $\psi(x) = 0$ (for any $x \in M$); and so $\psi = 0$. ■

Example. (Surjective is not necessarily sent to surjective) Notice that $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}$ is surjective; but $h^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}) \xrightarrow{h^{\mathbb{Z}}(\pi)} h^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is not surjective: $h^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ ($\mathbb{Z}$ has no torsion element) and $h^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \neq 0$.

Theorem 3 (Left exactness) *Suppose M is a (left) A-module and $0 \to N_1 \xrightarrow{\phi_1} N_2 \xrightarrow{\phi_1} N_3 \to 0$ is a S.E.S.; then*

$$0 \to h^M(N_1) \xrightarrow{h^M(\phi_1)} h^M(N_2) \xrightarrow{h^M(\phi_1)} h^M(N_3)$$

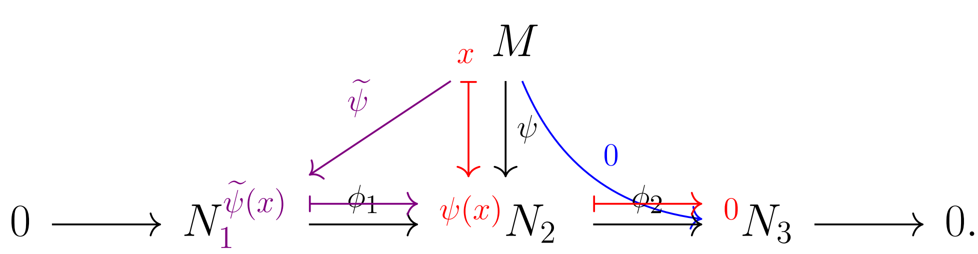
is an exact sequence.

Proof. By the previous lemma we know $h^M(\phi_1)$ is injective. So it is enough to show $\operatorname{Im} h^M(\phi_1) = \ker h^M(\phi_2)$. Since h^M is a functor and $\phi_2 \circ \phi_1 = 0$, we have

$$h^M(\phi_2) \circ h^M(\phi_1) = h^M(\phi_2 \circ \phi_1) = h^M(0) = 0;$$

and so $\operatorname{Im} h^M(\phi_1) \subseteq \ker h^M(\phi_2)$.

Suppose $\psi \in \ker h^M(\phi_2)$; that means $\phi_2 \circ \psi = 0$. Hence for any $x \in M$, $\psi(x) \in \ker \phi_2 = \operatorname{Im} \phi_1$. As ϕ_1 is injective, there is a unique element of N_1 that is mapped to $\psi(x)$; and so we get a function $\widetilde{\psi} : M \to N_1$ such that $\phi_1(\widetilde{\psi}(x)) = \psi(x)$.



Thus $\psi = h^M(\phi_1)(\widetilde{\psi}) \in \operatorname{Im} h^M(\phi_1)$; and claim follows. ■

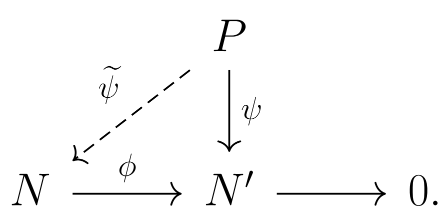
Next we find equivalent conditions of getting an exact functor; that means a functor that sends a S.E.S. to a S.E.S..

Theorem 4 (Projective modules) *Suppose P is a (left) A-module. Then the following statements are equivalent:*

(a) $h^P : A\text{-}\mathbf{mod} \to \mathbf{Ab}$ *is an exact functor.*

(b) *If* $\phi \in \mathrm{Hom}_A(N, N')$ *is surjective, then* $h^P(\phi)$ *is surjective.*

(c) *Suppose* $N \xrightarrow{\phi} N' \to 0$ *is exact. Then any* $\psi \in \mathrm{Hom}_A(P, N')$ *has a lift to* $\mathrm{Hom}_A(P, N)$*; that means there is* $\widetilde{\psi}$ *such that* $\phi \circ \widetilde{\psi} = \psi$*.*



(d) *A S.E.S. of the form* $0 \to M \to M' \to P \to 0$ *splits.*

(e) *P is a direct summand of a free module; that means there is a (left) A-module P' and a free (left) A-module F such that* $P \oplus P' \simeq F$*.*

A module P is called projective if the statements of the above theorem hold.

Remark. (1) Some books say P is projective if (c) holds; (2) The last property is the most hands on property of projective modules.

Proof of Theorem 4. By Theorem 3, h^P is a left exact functor. Hence we get that (a) $\Leftrightarrow$ (b). Notice that

$$\begin{aligned} \psi \in \operatorname{Im} h^P(\phi) &\Leftrightarrow \exists\, \widetilde{\psi} \in h^P(N), h^P(\phi)(\widetilde{\psi}) = \psi \\ &\Leftrightarrow \exists\, \widetilde{\psi} \in \operatorname{Hom}_A(P, N), \phi \circ \widetilde{\psi} = \psi; \end{aligned}$$

and so (b)$\Leftrightarrow$(c).
((c)$\Rightarrow$ (d)) By (c), id_P has a lift $\psi \in \operatorname{Hom}_A(P, M')$; and so the given S.E.S. splits;

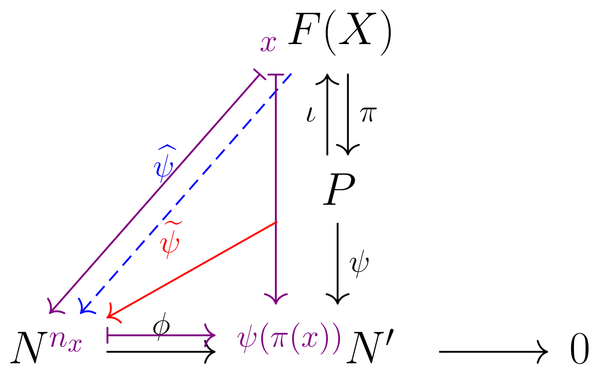
$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \overset{\psi}{\swarrow} & \big\downarrow \mathrm{id}_P & & \\ 0 & \longrightarrow & M & \longrightarrow & M' & \xrightarrow{\phi} & P & \longrightarrow & 0. \end{array}$$

((d)$\Rightarrow$ (e)) Let $F(P)$ be the free (left) A-module generated by the set P (here we are forgetting about the module structure of P). By the universal property of free modules, any function from P to a left A-module can be extended to a left A-module homomorphism from $F(P)$ to that module; we use this property for the identity function $\mathrm{id}_P : P \to P$. Hence we get a surjective A-module homomorphism $\phi : F(P) \to P$; and so the following is a S.E.S.

$$0 \to \ker\phi \to F(P) \to P \to 0.$$

By (d) this S.E.S. splits; and so $F(P) \simeq P \oplus \ker\phi$, and claim follows.

((e)$\Rightarrow$ (c)) By (e), there is a free A-module $F(X)$ and an A-module P' such that $\theta : F(X) \xrightarrow{\simeq} P \oplus P'$. Let $\pi : F(X) \to P$ be the projection to homomorphism induced by the projectionto the P-component; and $\iota : P \to F(X)$ be the embedding to the "first component" of $F(X)$ via θ. Then $\pi \circ \iota = \mathrm{id}_P$. Since ϕ is surjective, for any $x \in X$, there is $n_x \in N$ such that $\phi(n_x) = \psi(\pi(x))$. By the universal property of free modules, there is a unique A-module homomorphism $\widehat{\psi} : F(X) \to N$ such that, for any $x \in X$, $\widehat{\psi}(x) = n_x$. And so $\phi \circ \widehat{\psi}|_X = \psi \circ \pi|_X$; and since $F(X)$ is generated by X, we deduce that $\phi \circ \widehat{\psi} = \psi \circ \pi$. Let $\widetilde{\psi} := \widehat{\psi} \circ \iota$; then $\phi \circ \widetilde{\psi} = \phi \circ \widehat{\psi} \circ \iota = \psi \circ \pi \circ \iota = \psi$; and claim follows.



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