Math200b, lecture 11

Golsefidy

We have proved that a module is projective if and only if it is a direct summand of a free module; in particular, any free module is projective. For a given ring A, we would like to know to what extent the converse of this statement holds; and if it fails, we would like to somehow "measure" how much it does! In genral this is hard question; in your HW assignement you will show that for a local commutative ring A, any finitely generated projective module is free. By a result of Kaplansky the same statement holds for a module that is not necessarily finitely generated. Next we show that for a PID, any finitely generated projective module is free.

Proposition 1 *Let* D *be an integral domain. Then*

$$\textit{free } \mathrm{D}\textit{-module} \Rightarrow \textit{Projective} \Rightarrow \textit{torsion-free.}$$

If D *is a PID, for a finitely generated* D*-module all the above properties are equivalent.*

Proof. We have already discussed that a free module is projective. A projective module is a direct summand of a free module and a free module of an integral domain is torsion free. By the fundamental theorem of finitely generated modules over a PID, a torsion free finitely generated module over D is free; and claim follows. ■

Next we show $A = \mathbb{Z}[\sqrt{-10}]$ has a finitely generated projective module that is not free. In fact any ideal of A is projective; and since it is not a PID, it has an ideal that is not free. Based on the mentioned result of Kaplansky, a projective module is locally free. And for finitely generated modules, the converse of this statement holds as well: a finitely generated locally free module is projective. Hence by the previous proposition, if A is a Noetherian integral domain and $A_{\mathfrak{p}}$ is a PID for any $\mathfrak{p} \in \operatorname{Spec}(A)$, then any ideal of A is projective. In math200c, we will prove that $\mathbb{Z}[\sqrt{-10}]$ has this property (it is a Dedekind domain). For now, however, we present a hands-on approach and point out the connection with fractional ideals, having an inverse as a fractional ideal, and being projective.

Lemma 2 *Suppose* D *is an integral domian and* $\mathfrak{a} \trianglelefteq D$. *Then* $\mathfrak{a}$ *is a free* D*-module if and only if* $\mathfrak{a}$ *is a princpal ideal.*

Proof. ($\Rightarrow$) Since $\mathfrak{a}$ is a submodule of D, $\mathrm{rank}\mathfrak{a} \leq \mathrm{rank}D = 1$. So either $\mathrm{rank}\,\mathfrak{a} = 0$ or $\mathrm{rank}\,\mathfrak{a} = 1$. Since D has no zero-divisors, $\mathrm{rank}\,\mathfrak{a} = 0$ implies $\mathfrak{a} = 0$. Hence if a non-zero ideal is a free D-module, then it has rank 1; and so $\mathfrak{a} = aD$ for some $a \in D$. And claim follows.

($\Leftarrow$) Since $\mathfrak{a}$ is principal, $\mathfrak{a} = aD$ for some $a \in D$. If $a = 0$, then $\mathfrak{a} = 0$ is a free D-module. If $a \neq 0$, then $x \mapsto ax$ is a D-module isomorphism from D to $\mathfrak{a}$; and so $\mathfrak{a}$ is a free D-module. ■

Example. Let $A = \mathbb{Z}[\sqrt{-10}]$ and $\mathfrak{a} := \langle 2, \sqrt{-10}\rangle$. Then $\mathfrak{a}$ is not a free A-module.

Proof. Suppose to the contrary that $\mathfrak{a}$ is a free A-module; then by the previous lemma $\mathfrak{a}$ is a principal ideal. So $\mathfrak{a} = Aa$ for some a. So $a|2$ and $a|\sqrt{-10}$; hence $N(a)|\gcd(4, 10) = 2$. So $N(a)$ is either 1 or 2. Since there is no $x, y \in \mathbb{Z}$ such that $x^2 + 10y^2 = 2$, we deduce that $N(a) = 1$, which implies a is a unit; and so $\mathfrak{a} = A$. On the other hand, one can see that, for any $z \in \mathfrak{a}$, $\mathrm{N}(z)$ is even; and so $\mathfrak{a}$ is a proper ideal. Overall we get that $\mathfrak{a}$ is not a free A-module. ■

In the next lecture, we show that $\mathfrak{a}$ is a projective A-module.