Math200b, lecture 15

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Associativity of tensor product.

For a ring A, let ${}_A\mathbf{M}$ be the category of left A-modules. Suppose ${}_AM_B$ is an (A, B)-bimodule, and ${}_BN_C$ is a (B, C)-bimodule. Then tensoring by M and N give us functors $T_M : {}_B\mathbf{M} \to {}_A\mathbf{M}$ and $T_N : {}_C\mathbf{M} \to {}_B\mathbf{M}$; and so $T_M \circ T_N : {}_C\mathbf{M} \to {}_A\mathbf{M}$. We also notice that $M \otimes_B N$ is an (A, C)-bimodule; and so tensoring by $M \otimes_B N$ gives us a functor $T_{M\otimes_B N} : {}_C\mathbf{M} \to {}_A\mathbf{M}$. Next we show that these are essentially the same functors.

Theorem 1 *In the above setting there is a natural isomorphism*

$$\eta : T_M \circ T_N \xrightarrow{\sim} T_{M\otimes_B N}.$$

In fact, for any left C*-module* L*, there is a natural isomorphism*

$$\eta_L : M \otimes_B (N \otimes_C L) \longrightarrow (M \otimes_B N) \otimes_C L$$

such that $\eta_L(m \otimes (n \otimes l)) = (m \otimes n) \otimes l$*.*

Before we give a formal proof with more details, let us go over an alternative approach which is essentially behind the our formal argument as well:

Suppose M_i is an (A_i, A_{i+1})-bimodule for $i \in [1..n]$. Then there is a left A_1-module M and $f_0 : M_1 \times \cdots \times M_n \longrightarrow M$ such that the following property holds: suppose N is a left A_1-module and suppose

$$f : M_1 \times \cdots \times M_n \longrightarrow N$$

has the following properties: (1) A_1-linear in M_1, (2) linear in M_i for any i, and (3) A_i-balanced for $i \in [2..n-1]$. Then there is a unique A_1-module homomorphism $\phi_f : M \longrightarrow N$ such that $\phi_f \circ f_0 = f$, f_0 has the same properties as f, and M is generated by the image of f_0 as an A_1-module. One can show that any ordering of tensor products of M_i's satisfies the above universal property; in particular all of them are isomorphic as A_1-modules and one can check that is it a natural isomorphism.

Proof. Suppose L is a left C-module. For a given $m_0 \in M$, let

$$f_{m_0} : N \times L \to (M \otimes_B N) \otimes_C L, f_{m_0}(n, l) := (m_0 \otimes n) \otimes l.$$

One can check that f_{m_0} is linear in N and L, and C-balanced. Hence by the universal property of tensor product, there is an abelian group homomorphism

$$\phi_{m_0} : N \otimes_C L \to (M \otimes_B N) \otimes_C L, \phi_{m_0}(n \otimes l) = (m_0 \otimes n) \otimes l.$$

Now let

$$f : M \times (N \otimes_C L) \to (M \otimes_B N) \otimes_C N, f(m, x) := \phi_m(x);$$

in particular, $f(m, n \otimes l) = (m \otimes n) \otimes l$. Notice that f is linear in $N \otimes_C L$ as ϕ_m is an abelian group homomorphism. For any $n \in N$ and $l \in L$, we have

$$\begin{aligned} f(a_1 m_1 + a_2 m_2, n \otimes l) =& ((a_1 m_1 + a_2 m_2) \otimes n) \otimes l \\ =& (a_1(m_1 \otimes n) + a_2(m_2 \otimes n)) \otimes l \\ =& a_1((m_1 \otimes n) \otimes l) + a_2((m_2 \otimes n) \otimes l) \\ =& a_1 f(m_1, n \otimes l) + a_2 f(m_2, n \otimes l). \end{aligned}$$

Since f is linear in $N \otimes_C L$ and $N \otimes_C L$ is generated by $n \otimes l$'s as an abelian group, the above equality implies that f is A-linear in M.

For any $n \in N$ and $l \in L$, we have

$$\begin{aligned} f(m \cdot b, n \otimes l) =& ((m \cdot b) \otimes n) \otimes l \\ =& (m \otimes (b \cdot n)) \otimes l \\ =& f(m, (b \cdot n) \otimes l) \\ =& f(m, b \cdot (n \otimes l)). \end{aligned}$$

Since f is linear in $N \otimes_C L$, scalar multiplication by b is linear, and $N \otimes_C L$ is generated by $n \otimes l$'s as an abelian group, the above equality implies that f is B-balanced. Hence by the universal property of tensor product, there is an A-module homomorphism

$$\eta_L : M \otimes_B (N \otimes_C L) \to (M \otimes_B N) \otimes_C L, \eta_L(m \otimes (n \otimes l)) = (m \otimes n) \otimes l.$$

Similarly there is an A-module homomorphism

$$\lambda_L : (M \otimes_B N) \otimes_C L \to M \otimes_B (N \otimes_C L), \lambda_L((m \otimes n) \otimes l) = m \otimes (n \otimes l).$$

As pure tensor elements generate tensor products as abelian groups, the above equalities imply that η_L and λ_L are inverse of each other; and so η_L is an A-module isomorphism. It is easy to check that η_L's give us a natural transformation. ■

An immediate consequence of the above theorem is the following:

Proposition 2 *Suppose* M_B *is a flat right* B*-module and* ${}_BN_C$ *is a* (B, C)*-bimodule and a flat right* C*-module. Then* $M \otimes_B N$ *is a flat right* C*-module. In particular if* A *is a commutative ring and* M *and* N *are two flat* A*-modules, then* $M \otimes_A N$ *is a flat* A*-module.*

Proof. Since M_B is a flat right B-module, $T_M : {}_B\mathbf{M} \to \mathbf{Ab}$ is an exact functor. Since ${}_BN_C$ is a flat right C-module, $T_N : {}_C\mathbf{M} \to {}_B\mathbf{M}$ is an exact functor. And so $T_M \circ T_N : {}_C\mathbf{M} \to \mathbf{Ab}$ is an exact functor. By the above theorem there is a natural isomorphism $\eta : T_M \circ T_N \to T_{M\otimes_B N}$; and so $T_{M\otimes_B N}$ is an exact functor, which implies that $M \otimes_B N$ is a flat right C-module.

For a commutative ring A, a left (or right) A-module is an (A, A)-bimodule; and so the claim follows. ■

Tensor product and direct sum

Next we show that tensor product commutes with tensor product. In your HW assignment you will see how using the fact that there is a natural isomorphism $\prod_{i\in I} h^{M_i} \simeq h^{\oplus_{i\in I} M_i}$, one can show $h^{\oplus_{i\in I}(M_i\otimes_A N)} \simeq h^{(\oplus_{i\in I} M_i)\otimes_A N}$; this implies that

$$\bigoplus_{i\in I}(M_i \otimes_A N) \simeq (\bigoplus_{i\in I} M_i) \otimes_A N.$$

Here we prove this result for the case where the index set has two elements.

Proposition 3 *Suppose* ${}_AM_B$ *is an* (A, B)*-bimodule and* ${}_BN_1$ *and* ${}_BN_2$ *are two left* B*-modules. Then the following is a commutating diagram and* f *is an isomorphism of left* A*-modules.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M\otimes_B N_1 & \xrightarrow{id_M\otimes j_1} & M\otimes_B (N_1\oplus N_2) & \xrightarrow{id_M\otimes p_2} & M\otimes_B N_2 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & M\otimes_B N_1 & \xrightarrow{j_1} & (M\otimes_B N_1)\oplus (M\otimes_B N_2) & \xrightarrow{p_2} & M\otimes_B N_2 & \longrightarrow & 0
\end{array}$$

where $f(m\otimes(n_1, n_2)) = (m\otimes n_1, m\otimes n_2)$*; in particular the first row is a S.E.S..*

Proof. Let $l : M \times (N_1 \oplus N_2) \to (M \otimes_B N_1) \oplus (M \otimes_B N_2)$,

$$l(m, (n_1, n_2)) := (m \otimes n_1, m \otimes n_2).$$

It is easy to see that l is linear in both factors, B-balanced, and A-linear in the first factor. So using the universal property of tensor product, there is an A-module homomorphism

$$f : M \otimes_B (N_1 \oplus N_2) \to (M \otimes_B N_1) \oplus (M \otimes_B N_2), \text{such that}$$

$$f(m \otimes (n_1, n_2)) = (m \otimes n_1, m \otimes n_2).$$

Let

$$g : (M \otimes_B N_1) \oplus (M \otimes_B N_2) \to M \otimes_B (N_1 \oplus N_2),$$

$$g(x_1, x_2) := (\mathrm{id}_M \otimes j_1)(x_1) + (\mathrm{id}_M \otimes j_2)(x_2);$$

then g is a left A-module homomorphism, and

$$\begin{aligned} g(f(m \otimes (n_1, n_2)) =& (\mathrm{id}_M \otimes j_1)(m \otimes n_1) + (\mathrm{id}_M \otimes j_2)(m \otimes n_2) \\ =& m \otimes (n_1, 0) + m \otimes (0, n_2) \\ =& m \otimes (n_1, n_2); \end{aligned}$$

and so $g \circ f$ is identity. And

$$\begin{aligned} f(g(m_1 \otimes n_1, m_2 \otimes n_2)) =& f((\mathrm{id}_M \otimes j_1)(m_1 \otimes n_1) + (\mathrm{id}_M \otimes j_2)(m_2 \otimes n_2)) \\ =& f(m_1 \otimes (n_1, 0) + m_2 \otimes (0, n_2)) \\ =& (m_1 \otimes n_1, 0) + (0, m_2 \otimes n_2) \\ =& (m_1 \otimes n_1, m_2 \otimes n_2); \end{aligned}$$

and so $f \circ g$ is identity. Therefore f is an A-module isomorphism. It is easy to see that the above mentioned diagram is commuting. Hence the first row is isomorphic to the second row; and so it is a S.E.S.. ■

Proposition 4 *Suppose* M_B *and* M'_B *are two right* B*-modules; then* M *and* M' *are flat right* B*-modules if and only if* $M \oplus M'$ *is a flat right* B*-module.*

Proof. Suppose $f : N \to N'$ is an injective homomorphism of left B-modules. We get the following commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \otimes_B N & \xrightarrow{j_1 \otimes \mathrm{id}_N} & (M \oplus M') \otimes_B N & \xrightarrow{p_2 \otimes \mathrm{id}_N} & M' \otimes_B N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_M \otimes f} & & \downarrow{\scriptstyle \mathrm{id}_{M \oplus M'} \otimes f} & & \downarrow{\scriptstyle \mathrm{id}_{M'} \otimes f} & & \\ 0 & \longrightarrow & M \otimes_B N' & \xrightarrow{j_1 \otimes \mathrm{id}_{N'}} & (M \oplus M') \otimes_B N' & \xrightarrow{p_2 \otimes \mathrm{id}_{N'}} & M' \otimes_B N' & \longrightarrow & 0 \end{array}$$

By the previous proposition, each row is a S.E.S.. If M and M'

are flat, then $\mathrm{id}_M \otimes f$ and $\mathrm{id}_{M'} \otimes f$ are injective; and so by the Short Five Lemma, $\mathrm{id}_{M \oplus M'} \otimes f$ is injective, which implies that $M \oplus M'$ is flat.

If $M \oplus M'$ is flat, then $\mathrm{id}_{M\oplus M'} \otimes f$ is injective; and so by the above commuting diagram we have that $\mathrm{id}_M \otimes f$ is injective, which implies that M is flat. By symmetry, we deduce that M' is also flat; and claim follows. ■

Proposition 5 *A free left A-module F is flat.*

Proof. First we notice that we have proved earlier that $f_N : N \to N \otimes_A A$, $f(n) := n \otimes 1$ is an isomorphism of right A-modules. If $\phi : N \to N'$ is an injective right A-module homomorphism, then we have the following commuting diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad \phi \quad} & N' \\ \downarrow{\scriptstyle f_N} & & \downarrow{\scriptstyle f_{N'}} \\ N \otimes_A A & \xrightarrow{\phi \otimes \mathrm{id}_A} & N' \otimes_A A \end{array}$$

And so $\phi \otimes \mathrm{id}_A$ is injective, which implies that A is a flat A-module. Therefore, by the previous proposition and induction on n, A^n is a flat A-module.

Next we consider the general case; that means we can assume that $F = \bigoplus_{i \in I} A$ for some non-empty index set I. Suppose $\phi : N \to N'$ is an injective right A-module homomorphism. We have to show that $\phi \otimes \mathrm{id}_F : N \otimes_A F \to N' \otimes_A F$ is injective. Suppose that $x := \sum_{j=1}^{n} n_j \otimes f_j \in \ker(\phi \otimes \mathrm{id}_F) = 0$. So there is a finite subset J of I such that

$$f_1, \ldots, f_n \in \bigoplus_{j \in J} A;$$

here we are viewing $\bigoplus_{j \in J} A$ as a submodule of F. Notice that F can be viewed as an internal direct sum of $F_J := \bigoplus_{j \in J} A$ and $F_{I \setminus J} := \bigoplus_{i \in I \setminus J} A$. We have the following commuting diagram and by Proposition 3 each row is a S.E.S.:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \otimes_A F_J & \xrightarrow{\mathrm{id}_N \otimes i} & N \otimes_A F & \longrightarrow & N \otimes_A F_{I \setminus J} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' \otimes_A F_J & \xrightarrow{\mathrm{id}_{N'} \otimes i} & N' \otimes_A F & \longrightarrow & N' \otimes_A F_{I \setminus J} & \longrightarrow & 0
\end{array}$$

Let $x' := \sum_{j=1}^{n} n_j \otimes f_j \in N \otimes_A F_J$. So we have

$$\begin{aligned}
((\mathrm{id}_{N'} \otimes i) \circ (\phi \otimes \mathrm{id}_{F_J}))(x') &= (\phi \otimes \mathrm{id}_F) \circ (\mathrm{id}_N \otimes i))(x') \\
&= (\phi \otimes \mathrm{id}_F)(x) = 0.
\end{aligned}$$

Using the above diagram, and the fact that F_J is a flat A-module, we have that $x' = 0$; and so $x = 0$. ■

Theorem 6 *A projective left A-module P is flat.*

Proof. Since P is projective, it is a direct summand of a free A-module; that means there is a left A-module K such that $P \oplus K = F$ is a free left A-module. Suppose $\phi : N \to N'$ is an injective right A-module homomorphism. So the following is a commuting diagram and by Proposition 3 each row is a S.E.S.:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_A P & \xrightarrow{id_N \otimes i} & N \otimes_A F & \longrightarrow & N \otimes_A K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi \otimes id_P} & & \downarrow{\scriptstyle \phi \otimes id_F} & & \downarrow & & \\ 0 & \longrightarrow & N' \otimes_A P & \xrightarrow{id_{N'} \otimes i} & N' \otimes_A F & \longrightarrow & N' \otimes_A K & \longrightarrow & 0 \end{array}$$

By the previous proposition, $\phi \otimes id_F$ is injective. Hence by the above diagram, we deduce that $\phi \otimes id_P$ is injective; and so P is flat. ■

Algebras

Suppose A is a ring and $f : A \to R$ is a ring homomorphism such that $f(A) \subseteq Z(R)$; then we say R is an A-algebra. For instance any unital ring R can be viewed as a $\mathbb{Z}$-algebra as we have the ring homomorphism $f : \mathbb{Z} \to R, f(n) := n1_R$ and $f(\mathbb{Z}) \subseteq Z(R)$. Notice that an A-algebra is an (A, A)-bimodule. If R and S are two A-algebras, then $R \otimes_A S$ is an (A, A)-bimodule. Next theorem says that we can make $R \otimes_A S$ into an A-algebra.

Theorem 7 *Suppose R and S are two A-algebras; then the following gives us a well-defined operation on $R \otimes_A S$:*

$$(r \otimes s)(r' \otimes s') := rr' \otimes ss'$$

for any $r, r' \in R$ and $s, s' \in S$. This operation makes $R \otimes_A S$ a ring; and $f : A \to R \otimes_A S, f(a) := a(1 \otimes 1)$ makes $R \otimes_A S$ an A-algebra.

Instead of going through proof of this statement, in the next lecture, we will give some examples on how one can understand the algebra structure of tensor product of certain algebras.