1 Homework 5.

1. In this problem, you see the differences between a direct product and a direct sum. Among other things, you see that an infinite direct product is not necessarily a free module.

   (a) Let $\phi \in \mathrm{Hom}(\prod_{i=1}^{\infty} \mathbb{Z}, \mathbb{Z})$; let $e_j \in \prod_{i=1}^{\infty} \mathbb{Z}$ be

   $$e_j(i) := 0 \text{ if } i \neq j \quad \text{and } e_i(i) = 1.$$

   Suppose $\phi(e_j) = n_j \neq 0$ for every j. Choose a sequence of positive integers $1 =: k_1 < k_2 < \cdots$ such that

   $$k_{j+1} \nmid k_j! n_j. \tag{1}$$

   Consider

   $$\Sigma := \{(a_i)_{i=1}^{\infty} |\, a_i \in \{0, k_i!\}\}. \tag{2}$$

   (a-1) Argue why there exist two distinct elements $(a_i)_{i=1}^{\infty}$ and $(a'_i)_{i=1}^{\infty}$ of Σ such that

   $$\phi((a_i)_{i=1}^{\infty}) = \phi((a'_i)_{i=1}^{\infty}). \tag{3}$$

   (**Hint**. Notice that Σ is uncountble and $\mathbb{Z}$ is countable.)

   (a-2) In the setting of the previous step, suppose i_0 is the first index where $a_{i_0} \neq a'_{i_0}$. Show that

   $$\phi((a_{i_0} - a'_{i_0})e_{i_0}) \notin k_{i_0+1}\mathbb{Z}, \qquad \text{(\textbf{Hint}. use (1))}$$

   and

   $$\phi((a_{i_0} - a'_{i_0})e_{i_0}) \in k_{i_0+1}\mathbb{Z}; \qquad \text{(\textbf{Hint}. use (2) and (3))}$$

   and get a contradiction.

   (b) Use part (a) to deduce

   $$\mathrm{Hom}(\textstyle\prod_{i=1}^{\infty} \mathbb{Z}, \mathbb{Z}) \to \displaystyle\bigoplus_{i=1}^{\infty} \mathbb{Z},$$
   $$\phi \mapsto (\phi(e_i))_{i=1}^{\infty}$$

   is an isomorphism.

(**Hint**. Suppose $\bigoplus_{i=1}^{\infty}\mathbb{Z}\subseteq\ker\phi$; then show

$$p^n|\phi(pa_1,p^2a_2,p^3a_3,\ldots)$$

for every n and deduce that $\phi(pa_1,p^2a_2,p^3a_3,\ldots)=0$; observe that every element $(b_1,b_2,\ldots)$ can be written as a sum of two elements of the form $(2a_1,2^2a_2,\ldots)$ and $(3a_1,3^2a_2,\ldots)$.)

(c) Use part (b) to show $\prod_{i=1}^{\infty}\mathbb{Z}$ is not a free abelian group.

(d) Use part (b) to show

$$\mathrm{Hom}\left(\frac{\prod_{i=1}^{\infty}\mathbb{Z}}{\bigoplus_{i=1}^{\infty}\mathbb{Z}},\mathbb{Z}\right)=0.$$

2. Suppose A is an integral domain. Show that every submodule of a finitely generated free A-module is a free A-module if and only if A is a PID.

3. Suppose $(A,\mathfrak{m})$ is a local unital commutative ring; that means $\mathrm{Max}(A)=\{\mathfrak{m}\}$.

(a) (Nakayama's lemma) Suppose M is a finitely generated A-module. Suppose $M=\mathfrak{m}M$ where

$$\mathfrak{m}M=\Big\{\sum_{i=1}^{n}a_ix_i|\ a_i\in\mathfrak{m},x_i\in M\Big\}.$$

Prove that $M=0$.

(**Hint**. Let $y_1,\ldots,y_d$ be a generating set of M. By assumption, $\exists a_{ij}\in\mathfrak{m}$ such that

$$y_i=\sum_{j=1}^{d}a_{ij}y_j.$$

Hence $(I-[a_{ij}])\begin{pmatrix}y_1\\ \vdots\\ y_d\end{pmatrix}=0$. Show that $I-[a_{ij}]\in\mathrm{GL}_d(A)$; and deduce $y_i=0$; and so $M=0$.)

(b) Suppose M is a finitely generated A-module. Prove that

$$d(M)=\dim_{\frac{A}{\mathfrak{m}}}\left(\frac{M}{\mathfrak{m}M}\right),$$

where $\frac{M}{\mathfrak{m}M}$ is viewed as a vector space over $\frac{A}{\mathfrak{m}}$.

(**Hint**. It is clear that $d(M) \geq \dim_{\frac{A}{\mathfrak{m}}}(\frac{M}{\mathfrak{m}M})$; now suppose

$$y_1 + \mathfrak{m}M, \ldots, y_d + \mathfrak{m}M$$

is an $\frac{A}{\mathfrak{m}}$-basis of $\frac{M}{\mathfrak{m}M}$, and let N be the submodule of M that is generated by y_i's. Use part (a) for $\frac{M}{N}$.)

(c) (f.g. projective $\Rightarrow$ locally free) Suppose P is a finitely generated projective A-module. Prove that P is free.

(**Hint**. Suppose $d(P) = d$; so there is a S.E.S.

$$0 \to N \to A^d \to P \to 0.$$

Since P is projective, we have that there is an A-module isomorphism $\phi : A^d \xrightarrow{\sim} P \oplus N$. Show that $\phi(\mathfrak{m}A^d) = \mathfrak{m}P \oplus \mathfrak{m}N$; and then use part (b).)

(**Remark**. This exercise implies that for an arbitrary unital commutative ring A, a finitely generated projective module P is locally free; that means for every $\mathfrak{p} \in \operatorname{Spec}(A)$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. The converse of this statement is true as well: a f.g. locally free module is projective.)

4. Suppose $\{f_i\}_{i\in I} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ is a family of polynomials. For every unital commutative ring A, let

$$F(A) := \{(a_1, \ldots, a_n) \in A^n \mid \forall i \in I, f_i(a_1, \ldots, a_n) = 0\}.$$

(a) Prove that F defines a functor from the category of unital commutative rings to the category of sets.

(b) Prove that there exists a natural isomorphism from F to a representable functor.

(**Hint.** Let $\mathfrak{a} := \langle f_i \mid i \in I \rangle$ and $R_0 := \mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{a}$. Show that the following is a *natural bijection* between $\operatorname{Hom}_{\mathrm{Rng}}(R_0, A)$ and $F(A)$:

$$\phi \mapsto (\phi(x_1 + \mathfrak{a}), \ldots, \phi(x_n + \mathfrak{a})).$$

Notice that the inverse of this map is given by the evaluation maps; for every $\mathbf{a} \in F(A)$, let

$$\phi_{\mathbf{a}}(f(x) + \mathfrak{a}) := f(\mathbf{a})$$

and argue why this is well-defined.)

5. Suppose P and P' are projective A-modules, and

$$0 \to K \to P \xrightarrow{f} M \to 0$$

and

$$0 \to K' \to P' \xrightarrow{f'} M \to 0$$

are short exact sequences of A-modules. Prove that

$$P \oplus K' \simeq P' \oplus K.$$

(**Hint**. Let $L := \{(x, x') \in P \oplus P' |\, f(x) = f'(x')\}$. Show that L is a submodule of $P \oplus P'$. Notice that the following diagram is commuting and each row and column is an exact sequence; and then use the assumption that P and P' are projective to deduce $L \simeq P \oplus K'$ and $L \simeq P' \oplus K$.)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker\pi' & \xrightarrow{\sim} & K' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\pi & \longrightarrow & L & \xrightarrow{\pi'} & P' & \longrightarrow & 0\\
 & & \downarrow \wr & & \downarrow \pi & & \downarrow f' & & \\
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{f} & M & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$