1 Homework 6.

1. (Yoneda's lemma) Suppose $\underline{C}$ is a locally small category, $a \in \mathrm{Obj}(\underline{C})$,

$$h_a : \underline{C} \to \underline{\mathrm{Set}}$$

is a representable functor, and $F : \underline{C} \to \underline{\mathrm{Set}}$ is a functor. Let $\mathrm{Nat}(h_a, F)$ be the class of all natural transformations from h_a to F.

   (a) Prove that the following is a bijection:

   $$\theta_a : \mathrm{Nat}(h_a, F) \to F(a), \quad \theta(\eta) := \eta_a(1_a).$$

   (b) For $f \in \mathrm{Hom}_{\underline{C}}(a, a')$, let $\widehat{f} : h'_a \to h_a$ be

   $$\widehat{f}_b(a' \xrightarrow{g} b) := a \xrightarrow{g \circ f} b.$$

   Prove that $\widehat{f}$ is a natural transformation.

   (c) Prove that θ_a is a natural bijection; that means, if $f \in \mathrm{Hom}_{\underline{C}}(a, a')$, then the following is a commuting diagram

   $$\begin{tikzcd} \mathrm{Nat}(h_a, F) \arrow[r, "\theta_a"] \arrow[d, "\psi(f)"'] & F(a) \arrow[d, "F(f)"] \\ \mathrm{Nat}(h_{a'}, F) \arrow[r, "\theta_{a'}"] & F(a') \end{tikzcd}$$

   where for every $b \in \mathrm{Obj}(\underline{C})$,

   $$\psi(f)(\eta) := \eta \circ \widehat{f}.$$

2. Suppose D is a local Noetherian integral domain.

   (a) Prove that every submodule of a finitely generated projective D-module is projective if and only if D is a PID.

   (b) Find a local Noetherian integral domain which is not a PID.

   (c) Show that a submodule of a finitely generated projective module is not necessarily projective.

   (**Hint.** Use two results from last week (1) submodule of a finitely generated free D-module is free if and only if D is a PID, (2) finitely generated projective modules of a local Noetherian ring are free.)

3. Suppose A is a unital commutative ring, and M is an A-module. Let

$$T_M : \underline{A-\text{mod}} \to \underline{A-\text{mod}}, \ T_M(N) := M \otimes_A N \quad \text{and} \quad T_M(f) := \mathrm{id}_M \otimes f,$$

for $f \in \mathrm{Hom}_A(N, N')$.

(a) Prove that T_M is a functor.

(b) Prove that there exists a natural isomorphism between $T_{M_1} \circ T_{M_2}$ and $T_{M_1 \otimes_A M_2}$.

(**Hint**. For $x_1 \in M_1$, let

$$f_{x_1} : M_2 \times N \to (M_1 \otimes_A M_2) \otimes N, \quad f_{x_1}(x_2, y) := (x_1 \otimes x_2) \otimes y.$$

Prove that f_{x_1} is A-bilinear. Deduce that there exists an A-module homomorphism

$$\phi_{x_1} : M_2 \otimes N \to (M_1 \otimes_A M_2) \otimes N,$$

such that

$$\phi_{x_1}(x_2 \otimes y) = (x_1 \otimes x_2) \otimes y.$$

Let

$$f : M_1 \times (M_2 \otimes N) \to (M_1 \otimes_A M_2) \otimes N, \quad f(x_1, z) := \phi_{x_1}(z).$$

Prove that f is A-bilinear. Deduce that there exists an A-module homomorphism

$$\phi : M_1 \otimes_A (M_2 \otimes_A N) \to (M_1 \otimes_A M_2) \otimes N,$$

such that

$$\phi(x_1 \otimes z) = f(x_1, z);$$

in particular, for every $x_1 \in M_1$, $x_2 \in M_2$, and $y \in N$,

$$\phi(x_1 \otimes (x_2 \otimes y)) = (x_1 \otimes x_2) \otimes y.$$

Similarly there exists an A-module homomorphism

$$\psi : (M_1 \otimes_A M_2) \otimes N \to M_1 \otimes_A (M_2 \otimes_A N),$$

such that

$$\psi((x_1 \otimes x_2) \otimes y) = x_1 \otimes (x_2 \otimes y).$$

Deduce that ϕ and ψ are inverse of each other. Convince yourself that this is a natural isomorphism. You do not need to include that in your solution.)

(**Remark**. We say that $M_1 \otimes_A (M_2 \otimes_A N)$ and $(M_1 \otimes_A M_2) \otimes_A N$ are naturally isomorphic.)

4. (You do not have to write anything for this problem; only justify and understand all the statements. This is a useful result to have in your toolbox.)

   For two functors F_1 and F_2, we say $F_1 \simeq F_2$ if there exists a natural isomorphism $\eta : F_1 \to F_2$. Suppose $\{M_i\}_{i\in I}$ is a family of A-modules and N is an A-module.

   (a) Suppose $\{F_i\}_{i\in I}$ is a family of functors from $\underline{A\text{-mod}}$ to itself. Define the functor $\prod_{i\in I} F_i$.

   (b) Prove that
   $$\prod_{i\in I} h_{M_i} \simeq h_{\bigoplus_{i\in I} M_i}.$$

   (c) Justify why we have
   $$\begin{aligned} h_{\bigoplus_{i\in I} N\otimes_A M_i} \simeq \prod_{i\in I} h_{N\otimes_A M_i} &\simeq \prod_{i\in I}(h_{M_i} \circ h_N) \\ &\simeq (\prod_{i\in I} h_{M_i}) \circ h_N \simeq h_{\bigoplus_{i\in I} M_i} \circ h_N \\ &\simeq h_{N\otimes_A (\bigoplus_{i\in I} M_i)}. \end{aligned}$$

   (d) Prove that $\bigoplus_{i\in I}(N \otimes_A M_i) \simeq N \otimes_A (\bigoplus_{i\in I} M_i)$ as A-modules.