1 Homework 7.

1. Suppose A is a local unital commutative ring and K is a field.

   (a) Suppose V and W are two K-vector spaces. Prove that

   $$\dim_K(V \otimes_K W) = (\dim_K V)(\dim_K W)$$

   (**Hint**. Use problem 5.)

   (b) Suppose M and N are finitely generated A-modules, and $M \otimes_A N = 0$. Prove that either $M = 0$ or $N = 0$.

   (Hint. Suppose $\mathrm{Max}(A) = \{\mathfrak{m}\}$. Let $k := A/\mathfrak{m}$. Argue

   $$M/\mathfrak{m}M \simeq M \otimes_A k \quad \text{and} \quad N/\mathfrak{m}N \simeq N \otimes_A k.$$

   Show that $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) = 0$. Use Nakayama's lemma.)

   (**Remark**. Notice that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$ and so it is crucial that A is local. For an arbitrary ring A, we deduce that $M \otimes_A N = 0$ implies for any $\mathfrak{p} \in \operatorname{Spec} A$ either $M_{\mathfrak{p}} = 0$ or $N_{\mathfrak{p}} = 0$.)

2. Suppose A is a unital commutative ring, $S \subseteq A$ is a multiplicatively closed subset, and M is an A-module.

   (a) Convince yourself that localizing defines an exact functor from $\underline{A\text{-mod}}$ to $\underline{S^{-1}A\text{-mod}}$. (You do not need to write any argument for this part.)

   (b) Prove that $S^{-1}A \otimes_A M \simeq S^{-1}M$; deduce that $S^{-1}A$ is a flat A-module.

   (c) Prove that, if M is a flat A-module, then $S^{-1}M$ is a flat $S^{-1}A$-module.

   (d) Prove that $\frac{x_1 \otimes x_2}{1} \mapsto \frac{x_1}{1} \otimes \frac{x_2}{1}$ gives us a well-defined $S^{-1}A$-module isomorphism

   $$S^{-1}(M_1 \otimes_A M_2) \xrightarrow{\sim} S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2.$$

   (e) Prove that, if $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \operatorname{Spec}(A)$, then M is flat. (Hint: look at previous HWs on localizing a module.)

3. Suppose M_i's and N are A-modules, M_3 is flat, and
$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$
is a SES. Prove that
$$0 \to M_1 \otimes_A N \xrightarrow{f_1 \otimes \mathrm{id}_N} M_2 \otimes_A N \xrightarrow{f_2 \otimes \mathrm{id}_N} M_3 \otimes_A N \to 0$$
is a SES.

(**Hint.** Argue that there exists a SES
$$0 \to K \to F \to N \to 0$$
such that F is a free A-module. Discuss why the following is a commutating diagram where all the rows and columns are exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M_1 \otimes_A N & \longrightarrow & M_2 \otimes_A N & \longrightarrow & M_3 \otimes_A N & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_1 \otimes_A F & \longrightarrow & M_2 \otimes_A F & \longrightarrow & M_3 \otimes_A F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M_3 \otimes_A K & \longrightarrow & M_3 \otimes_A K & \longrightarrow & M_3 \otimes_A K & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

Suppose $x \in M_1 \otimes_A N$ is in the kernel of $f_1 \otimes \mathrm{id}_N$. Then use the following diagram:

$$\begin{array}{ccccc}
x & \longmapsto & 0 & & \\
\uparrow & & \uparrow & & \\
y, y' & \rightrightarrows & z & \longmapsto & 0 \\
\uparrow & & \uparrow & & \uparrow \\
w & \longmapsto & u & \longmapsto & ?= 0
\end{array}$$

Start with red, deduce existence of green, get the violet part, continue with blue. Argue why $y = y'$; and deduce that $x = 0$.)

4. Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is a SES of A-modules and M_3 is flat. Prove that M_1 is flat if and only if M_2 is flat.

(**Hint.** Use the previous problem and the Short Five Lemma.)