1 Homework 9.

1. Suppose M_i's and N are A-modules, M_3 is flat, and

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

is a SES. Prove that

$$0 \to M_1 \otimes_A N \xrightarrow{f_1 \otimes \mathrm{id}_N} M_2 \otimes_A N \xrightarrow{f_2 \otimes \mathrm{id}_N} M_3 \otimes_A N \to 0$$

is a SES.

(**Hint.** Argue that there exists a SES

$$0 \to K \to F \to N \to 0$$

such that F is a free A-module. Discuss why the following is a commutating diagram where all the rows and columns are exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M_1 \otimes_A N & \longrightarrow & M_2 \otimes_A N & \longrightarrow & M_3 \otimes_A N & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_1 \otimes_A F & \longrightarrow & M_2 \otimes_A F & \longrightarrow & M_3 \otimes_A F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M_3 \otimes_A K & \longrightarrow & M_3 \otimes_A K & \longrightarrow & M_3 \otimes_A K & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

Suppose $x \in M_1 \otimes_A N$ is in the kernel of $f_1 \otimes \mathrm{id}_N$. Then use the following diagram:

$$\begin{array}{ccccc}
x & \longmapsto & 0 & & \\
\uparrow & & \uparrow & & \\
y, y' & \rightrightarrows & z & \longmapsto & 0 \\
\uparrow & & \uparrow & & \uparrow \\
w & \longmapsto & u & \longmapsto & ?= 0
\end{array}$$

Start with red, deduce existence of green, get the violet part, continue with blue. Argue why $y = y'$; and deduce that $x = 0$.)

2. Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is a SES of A-modules and M_3 is flat. Prove that M_1 is flat if and only if M_2 is flat.

   (**Hint.** Use the previous problem and the Short Five Lemma.)

3. Suppose E/F is a field extension, $\alpha \in E$, and $[F[\alpha] : F]$ is odd. Prove that $F[\alpha^2] = F[\alpha]$.

4. Suppose $a_1, \ldots, a_n$ are positive rational numbers. Prove that $\sqrt[3]{2}$ is not in $\mathbb{Q}[\sqrt{a_1}, \ldots, \sqrt{a_n}]$.

5. Suppose $E \subseteq \mathbb{C}$ is a splitting field of $x^p - 2$ over $\mathbb{Q}$ where p is an odd prime number.

   (a) Prove that $E = \mathbb{Q}[\sqrt[p]{2}, \zeta_p]$ where $\zeta_p := e^{2\pi i/p}$.

   (b) Prove that $[E : \mathbb{Q}] = p(p-1)$.

   .(**Hint.** Notice that $[E : \mathbb{Q}]$ is a multiple of $[\mathbb{Q}[\zeta_p] : \mathbb{Q}]$ and $[\mathbb{Q}[\sqrt[p]{2}] : \mathbb{Q}]$. Argue that $[\mathbb{Q}[\zeta_p] : \mathbb{Q}] = p - 1$ and $[\mathbb{Q}[\sqrt[p]{2}] : \mathbb{Q}] = p$.)

6. Suppose E is a splitting field of $f(x) \in F[x]$ over F.

   (a) Prove that if $\gcd(f, f') \neq 1$, then $E \otimes_F F[x]/\langle f \rangle$ has a non-zero nilpotent element.

   (b) Prove that if $\gcd(f, f') = 1$, then

$$E \otimes_F (F[x]/\langle f \rangle) \simeq \underbrace{E \oplus \cdots \oplus E}_{\deg f\text{-times}}.$$

7. Suppose p is an odd prime, and $a \in \mathbb{F}_p^{\times}$. Prove that $x^p - x + a$ is irreducible in $\mathbb{F}_p[x]$.

   (**Hint.** Let E be a splitting field of $x^p - x + a$ over $\mathbb{F}_p$. Let $\alpha \in E$ be a zero of $x^p - x + a$. Prove that $\alpha + i$ is a zero of $x^p - x + a$ for every $i \in \mathbb{F}_p$, and deduce that

$$x^p - x + a = \prod_{i \in \mathbb{F}_p} (x - \alpha - i).$$

   Notice that $m_{\alpha, \mathbb{F}_p}(x)$ divides $x^p - x + a$, and consider the coefficient of x^{d-1} in $m_{\alpha, \mathbb{F}_p}(x)$ and show that $\deg m_{\alpha, \mathbb{F}_p} = p$.)