Homework 2

Friday, April 13, 2018 2:22 PM

1. (a) Suppose $\forall \mathfrak{p} \in \mathrm{Spec}(A)$, $\mathrm{Nil}(A_{\mathfrak{p}})=0$. Prove that $\mathrm{Nil}(A)=0$.

(Hint. Suppose $x \in \mathrm{Nil}(A)\setminus\{0\}$. Consider $(0:x)$.)

(b) Suppose B is a boolean ring; that means $a^2=a$ $\forall a \in B$.

(b-1) Prove that $\mathrm{Spec}(B)=\mathrm{Max}(B)$ and $\forall \mathfrak{p} \in \mathrm{Spec}(B)$, $B/\mathfrak{p} \simeq \mathbb{Z}/2\mathbb{Z}$

(b-2) Prove that, $\forall \mathfrak{p} \in \mathrm{Spec}(B)$, $B_{\mathfrak{p}} \simeq \mathbb{Z}/2\mathbb{Z}$

(Hint. If $a \in \mathfrak{p}$, then $1-a \notin \mathfrak{p}$ and $\frac{a}{1}=\frac{a(1-a)}{1-a}=0$.)

(b-3) Let $B=P(X)$ be the power set of X; and let $b_1+b_2:=b_1 \triangle b_2$ be the symmetric difference and $b_1 \cdot b_2 := b_1 \cap b_2$. Convince yourself that $(B,+,\cdot)$ is a boolean ring. Prove that B is Noetherian if and only if $|X|<\infty$.

(c) Suppose, $\forall \mathfrak{p} \in \mathrm{Spec}(A)$, $A_{\mathfrak{p}}$ is an integral domain;

Is A necessarily an integral domain?

(d) Suppose, $\forall \mathfrak{p} \in \mathrm{Spec}(A)$, $A_{\mathfrak{p}}$ is Noetherian; Is A necessarily Noetherian?

Homework 2

Friday, April 13, 2018 2:50 PM

2. Let B be a flat A-algebra. Then the following statements are equivalent:

(i) $\mathfrak{a}^{ec} = \mathfrak{a}$ for any $\mathfrak{a} \trianglelefteq A$.

(ii) $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is surjective.

(iii) $\forall\, \mathfrak{m} \in \mathrm{Max}(A)$, $\mathfrak{m}^e \neq B$.

(iv) If M is any non-zero A-module, then $M \otimes_A B \neq 0$.

(v) For every A-module M, $\begin{aligned} M &\longrightarrow M \otimes_A B \\ x &\longmapsto x \otimes 1 \end{aligned}$ is injective.

(Hint. (i) $\Rightarrow$ (ii); we proved $\mathfrak{p} \in \mathrm{Im}\, f^* \Leftrightarrow \mathfrak{p} = \mathfrak{p}^{ec}$;

(ii) $\Rightarrow$ (iii); is clear;

(iii) $\Rightarrow$ (iv); For any $x \in M$, $0 \to Ax \to M$ is exact

Since B is a flat A-module, $0 \to Ax \otimes_A B \to M \otimes_A B$ is exact.

So it is enough to show $Ax \otimes_A B \neq 0$. Notice $Ax \simeq A/\mathfrak{a}$

and $A/\mathfrak{a} \otimes_A B \simeq B/\mathfrak{a}^e$.

(iv) $\Rightarrow$ (v); Suppose $M' = \ker(M \to M \otimes_A B)$. Since B is a flat A-mod, $0 \to M' \otimes_A B \to M \otimes_A B \xrightarrow[f]{} (M \otimes_A B) \otimes_A B$ is exact. Let $g : \underbrace{(M \otimes_A B)}_{B\text{-mod}} \otimes_A B \to M \otimes_A B$, $g(x \otimes b) := xb$. Show that g is a well-defined B-mod. hom, and $g \circ f = \mathrm{id}$. And so f is injective.

(v) $\Rightarrow$ (i) Show $A/\mathfrak{a} \hookrightarrow B/\mathfrak{a}^e$.)

We say B is faithfully flat over A if these properties hold.

Homework 2

Friday, April 13, 2018 3:28 PM

3. (a) Let $A = k[x,y,z]/\langle xy - z^2 \rangle$, $\mathfrak{p} := \langle \bar{x}, \bar{z} \rangle$. Prove that $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{p}^2$ is NOT primary.

(b) Let $\mathfrak{q} = \langle x, y^2 \rangle \triangleleft k[x,y]$. Prove $\sqrt{\mathfrak{q}} = \underbrace{\langle x, y \rangle}_{\mathfrak{m}} \in \operatorname{Max}(k[x,y])$;

Deduce that $\mathfrak{q}$ is $\mathfrak{m}$-primary. Show $\mathfrak{q} \neq \mathfrak{m}^n$ $(\forall n \in \mathbb{Z})$.

(c) Let $\mathfrak{a} = \langle x^2, xy \rangle \triangleleft k[x,y]$. Find two reduced primary decompositions for $\mathfrak{a}$.

(In this problem, k is a field; x, y, z are indeterminants.)

4. For $\mathfrak{a} \triangleleft A$, let $\mathfrak{a}[x] = \{ \sum_{i=0}^{m} a_i x^i \mid a_i \in \mathfrak{a}, m \in \mathbb{Z}^{\geq 0} \}$. Consider $A \hookrightarrow A[x]$.

(a) Convince yourself that $\mathfrak{a}^e = \mathfrak{a}[x]$. Show that $\operatorname{Spec}(A) \xhookrightarrow{e} \operatorname{Spec}(A[x])$ is a well-defined injection.

(b) Prove that, if $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of A, then $\mathfrak{q}^e$ is a $\mathfrak{p}^e$-primary ideal of $A[x]$.

(d) Prove that $0 \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots \subsetneq \langle x_1, \ldots, x_n \rangle$ is a chain of prime ideals of $k[x_1, \ldots, x_n]$ and $\langle x_1, \ldots, x_r \rangle^s$ is $\langle x_1, \ldots, x_r \rangle$-primary. ($k$: field and x_i's are indeter.)

Homework 2

Friday, April 13, 2018 4:12 PM

5. Suppose $\mathfrak{p} \in \mathrm{Spec}(A)$. Consider $A \longrightarrow A_{\mathfrak{p}}$; and let $\mathfrak{p}^{(n)} := (\mathfrak{p}^n)^{ec}$.

(i) Prove that $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary.

(ii) Prove that, if $\mathfrak{p}^n$ is decomposable, then $\mathfrak{p}$ is the smallest element of $\mathrm{Ass}(\mathfrak{p}^n)$; and $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-factor.

In particular, $\mathfrak{p}^n$ is primary if and only if $\mathfrak{p}^n = \mathfrak{p}^{(n)}$.