Homework 3

Wednesday, April 18, 2018 12:08 AM

1. Suppose any $\mathfrak{p} \in \operatorname{Spec}(A)$ is finitely generated. Prove that A is Noetherian.

(<u>Hint.</u> Let $\Sigma := \{\mathfrak{a} \lhd A \mid \mathfrak{a} \text{ is not finitely generated}\}$. Suppose $\Sigma \neq \emptyset$. Prove that Σ has a maximal element $\mathfrak{a}$. By assumption $\mathfrak{a}$ is not prime. And so $\exists\, x, y \notin \mathfrak{a}$ s.t. $xy \in \mathfrak{a}$.

- Show that $\exists\, a_1, \ldots, a_n \in \mathfrak{a}$ s.t. $\mathfrak{a} + \langle x \rangle = \langle a_1, \ldots, a_n, x \rangle$.
- Show that $\mathfrak{a} = \langle a_1, \ldots, a_n \rangle + (\mathfrak{a} : x)\, x$
- Show that $(\mathfrak{a} : x)$ is finitely generated
- Deduce that $\mathfrak{a}$ is finitely generated and get a contradiction.)

2. Suppose, for any $\mathfrak{m} \in \operatorname{Max}(A)$, $A_{\mathfrak{m}}$ is Noetherian and for any $x \in A \setminus \{0\}$, $\left|\{\mathfrak{m} \in \operatorname{Max}(A),\ \mathfrak{m} \mid \langle x \rangle\}\right| < \infty$. Prove that A is Noetherian.

(<u>Hint:</u> Suppose $0 \neq \mathfrak{a} \lhd A$. Then $|\{\mathfrak{m} \in \operatorname{Max}(A) \mid \mathfrak{m} \mid \mathfrak{a}\}| < \infty$; and for any $\mathfrak{m} \mid \mathfrak{a}$, $\exists$ a f.g. ideal $\mathfrak{a}(\mathfrak{m}) \subseteq \mathfrak{a}$ s.t. $\mathfrak{a}(\mathfrak{m})_{\mathfrak{m}} = \mathfrak{a}_{\mathfrak{m}}$.

Take $a \in \mathfrak{a} \setminus \{0\}$, and for any $\mathfrak{m} \nmid \mathfrak{a}$ and $\mathfrak{m} \mid \langle a \rangle$,

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Let $a(\mathfrak{m}) \in \mathfrak{a} \setminus \mathfrak{m}$. Consider

$$\mathfrak{a}' := \sum_{\mathfrak{m} \mid \mathfrak{a}} \mathfrak{a}(\mathfrak{m}) + \langle a \rangle + \sum_{\substack{\mathfrak{m} \nmid \mathfrak{a} \\ \mathfrak{m} \mid \langle a \rangle}} \langle a(\mathfrak{m}) \rangle \subseteq \mathfrak{a}.$$

Show that $\mathfrak{a}'_{\mathfrak{m}} = \mathfrak{a}_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Max}(A)$.)

3. Show that $\langle x,y \rangle^2$ is a primary ideal of the ring of polynomials $k[x,y]$ where k is a field, but $\langle x,y\rangle^2$ is not irreducible.

4. Height of $\mathfrak{p} \in \mathrm{Spec}(A)$ is defined to be

$$\mathrm{ht}(\mathfrak{p}) := \sup \{ n \in \mathbb{Z}^{\geq 0} \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p} \}.$$
$$\mathfrak{p}_i \in \mathrm{Spec}(A).$$

Later in the course we will prove Krull's principal ideal theorem: Suppose A is Noetherian, and $\mathfrak{p}$ is a minimal prime in $V(\langle a \rangle)$ where $a \notin A^{\times}$. Then $\mathrm{ht}(\mathfrak{p}) \leq 1$.

In this problem you can use Krull's Principal Ideal Theorem. Suppose A is a Noetherian integral domain. Prove that A is a UFD if and only if every prime ideal of height 1 is principal.

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5. For any A-module M, let $[M]$ be vaguely the class of all A-modules isomorphic to M. Let

$$\mathrm{Pic}(A) := \{[M] \mid M \text{ is a finitely generated projective } A\text{-module s.t. } \forall \mathfrak{p} \in \mathrm{Spec}(A),\ M_{\mathfrak{p}} \simeq A_{\mathfrak{p}}\}$$

(a) Show that, if $[M]$ and $[M'] \in \mathrm{Pic}(A)$, then $[M \otimes_A M'] \in \mathrm{Pic}(A)$.

(b) Show that, if $[M] \in \mathrm{Pic}(A)$, then $[M^*] \in \mathrm{Pic}(A)$ where $M^* := \mathrm{Hom}_A(M, A)$.

(c) Prove that $M^* \otimes_A M \simeq A$.

(d) Show that $\mathrm{Pic}(A)$ is a group where $[M_1] \cdot [M_2] := [M_1 \otimes M_2]$.

(This is called the Picard group of A).

6. (Gelfand–Kirillov dimension). In this problem, you will explore basic properties of $\mathrm{GKdim}(A)$. This combinatorial dimension tells us about the growth rate of an algebra, and works equally well for non-commutative rings. So here we assume tha A is a unital ring which is not necessarily commutative.

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Suppose A is a finitely generated k-algebra where k is a field. Let V be a finite dimensional subspace of A that contains 1 and a generating set of A. Then $0 \subseteq V \subseteq V\cdot V \subseteq V\cdot V\cdot V \subseteq \cdots$.

Let $\Pi_n V := \underbrace{V\cdot V\cdot \cdots \cdot V}_{n \text{ times}} = \{v_1 \cdots v_n \mid v_i \in V\}$; notice that $A = \bigcup_{n=1}^{\infty} (\Pi_n V)$. Let $d_V := \limsup_{n\to\infty} \dfrac{\log(\dim_k \Pi_n V)}{\log n}$.

ⓐ Suppose W is another finite dimensional subspace of A that contains 1 and generates A. Prove that $d_V = d_W$.

This value is called the Gelfand-Kirillov dimension of A; and it is denoted by GKdim A.

ⓑ Prove the following basic properties of GKdim A:

(b-1) Suppose $A \subseteq B$ are two finitely generated k-algebras. Prove that GKdim $A \leq$ GKdim B.

(b-2) Suppose $\mathfrak{a} \trianglelefteq A$, $B = A/\mathfrak{a}$, where A is a finitely generated k-algebra. Prove that GKdim $B \leq$ GKdim A.

(b-3) GKdim $A[x] =$ GKdim $A + 1$; and so GKdim $k[x_1, \ldots, x_n] = n$ ($A[x]$ and $k[x_1, \ldots, x_n]$ are rings of polynomials.)

ⓒ Suppose $A \subseteq B$ are finitely generated k-algebras and B is a finitely generated A-mod. Prove that GKdim $A =$ GKdim B.