Homework 4

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1. Suppose B/A is an integral extension. Prove that $J(A) = J(B) \cap A$

where $J(\cdot)$ is the Jacobson radical of $\cdot$.

2. (a) Let $B := \mathbb{Z}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}]$ be the ring of polynomials.

Let $z_i = z_i(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}) \in \mathbb{Z}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}]$ be

such that $(T^n + x_{n-1}T^{n-1} + \cdots + x_1 T + x_0)(T^m + y_{m-1}T^{m-1} + \cdots + y_0)$

$$= T^{n+m} + z_{n+m-1} T^{n+m-1} + \cdots + z_1 T + z_0 .$$

Prove that $\mathbb{Z}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}]$ is a finitely generated

$\mathbb{Z}[z_0, \ldots, z_{n+m-1}]$- module.

(b) Suppose B/A is a ring extension and C is the integral extension of A in B. Suppose $f, g \in B[x]$ are monic polynomials. Prove that

$f(x)g(x) \in C[x] \Leftrightarrow f(x), g(x) \in C[x]$.

(Hint. (a) Let F be the field of fractions of A and E/F be the splitting field of $T^{n+m} + z_{n+m-1}T^{n+m-1} + \cdots + z_0$ over F. Deduce all the zeros of $T^n + x_{n-1}T^{n-1} + \cdots + x_0$ and $T^m + y_{m-1}T^{m-1} + \cdots + y_0$ are integral over $\bar{A} := \mathbb{Z}[z_0, \ldots, z_{n+m-1}]$. (b) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and

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$g(x) = x^m + a'_{m-1} x^{m-1} + \cdots + a'_0$. Use part (a) and "evaluate" at $a_0, \ldots, a_{n-1}, a'_0, \ldots, a'_{m-1}$ to deduce the subring generated by a_i, a'_j's is integr. over the subring generated by the coeff. of $f(x)g(x)$.)

3. (a) Suppose B/C is a ring extension and $B \setminus C$ is closed under multiplication. Prove that C is integrally closed in B.

(b) Suppose B/A is a ring extension and C is the integral closure of A in B. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

(Hint (a) Suppose $b \in B$ is integral over C, and let $n \in \mathbb{Z}^+$ be the smallest number such that $b^n + c_{n-1} b^{n-1} + \cdots + c_0 = 0$ for some $c_i \in C$. And show, if $b \notin C$, then $n-1$ also satisfies the above property.

(b) Show $B[x] \setminus C[x]$ is closed under multiplication.)

4. (a) Suppose A is a ring and G is a finite subgroup of $\mathrm{Aut}(A)$. Let $A^G := \{a \in A \mid \forall \sigma \in G, \sigma(a) = a\}$. Prove that A/A^G is an integral extension.

(b) For $\mathfrak{p} \in \mathrm{Spec}(A^G)$, prove that $G \curvearrowright (f^*)^{-1}(\mathfrak{p})$ transitively

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where $f: A^G \hookrightarrow A$.

5. Suppose $k/\mathbb{Q}$ is a finite Galois extension. Let $\mathcal{O}_k$ be the integral closure of $\mathbb{Z}$ in k. Suppose $\mathcal{O}_k$ is a finitely generated ring. Prove that

(1) $\mathcal{O}_k$ is integrally closed.

(2) $\dim \mathcal{O}_k = 1$; and so $\operatorname{Spec} \mathcal{O}_k = \{0\} \cup \operatorname{Max}\{\mathcal{O}_k\}$.

(3) For any prime number p, $\{\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_k) \mid \mathfrak{p} \mid p\mathcal{O}_k\}$ is a non-empty finite set, and $\operatorname{Gal}(k/\mathbb{Q})$ acts transitively on this set.

(4) $\forall \mathfrak{a} \underset{\neq}{\trianglelefteq} \mathcal{O}_k$, $\exists!$ (up to permutation) primary ideals $\mathfrak{q}_i$ s.t.

$$\mathfrak{a} = \mathfrak{q}_1 \cdot \mathfrak{q}_2 \cdot \dots \cdot \mathfrak{q}_n.$$

(5) If $\mathfrak{q}$ is a non-zero primary ideal, then $\sqrt{\mathfrak{q}} =: \mathfrak{m} \in \operatorname{Max} \mathcal{O}_k$ and $\exists n \in \mathbb{Z}^+$, $\mathfrak{m}^n \subseteq \mathfrak{q}$.

(6) $\forall \mathfrak{m} \in \operatorname{Max}(\mathcal{O}_k)$, $\mathcal{O}_k/\mathfrak{m}$ is a finite field.

(7) If $\mathfrak{a} \triangleleft \mathcal{O}_k$, then $N_{k/\mathbb{Q}}(\mathfrak{a}) := |\mathcal{O}_k/\mathfrak{a}| < \infty$.