1. Suppose \( B/A \) is an integral extension. Prove that \( J(A) = J(B) \cap A \)
where \( J(\cdot) \) is the Jacobson radical of \( \cdot \).

2. (a) Let \( B = \mathbb{Z}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}] \) be the ring of polynomials.

Let \( z_i = z_i (x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}) \in \mathbb{Z}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}] \) be such that

\[
(T^n + x_{n-1} T^{n-1} + \cdots + x_1 T + x_0)(T^m + y_{m-1} T^{m-1} + \cdots + y_0) = T^{n+m} + z_{n+m-1} T^{n+m-2} + \cdots + z_1 T + z_0.
\]

Prove that \( \mathbb{Z}[x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}] \) is a finitely generated \( \mathbb{Z}[z_0, \ldots, z_{n+m-1}] \) module.

(b) Suppose \( B/A \) is a ring extension and \( C \) is the integral extension
of \( A \) in \( B \). Suppose \( f, g \in B[x_1] \) are monic polynomials. Prove that

\[ f(x) \mid g(x) \iff f(x), g(x) \in C[x_1]. \]

(Hint: (a) Let \( F \) be the field of fractions of \( A \) and \( E/f \) be the

splitting field of \( T^{n+m} + z_{n+m-1} T^{n+m-2} + \cdots + z_0 \) over \( F \). Deduce all the
zeros of \( T^n + x_{n-1} T^{n-1} + \cdots + x_0 \) and \( T^m + y_{m-1} T^{m-1} + \cdots + y_0 \) are integral over \( \overline{A} = \mathbb{Z}[z_0, \ldots, z_{n+m-1}] \).

(b) Let \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) and
$g(x) = x^n + a_{m-1}x^{m-1} + \cdots + a_0$. Use part (a) and "evaluate" at $a_0, \ldots, a_{m-1}$, $a_0', \ldots, a_{m-1}'$ to deduce the subring generated by $a_i, a_i'$'s is integral over the subring generated by the coeff. of $x^n$ in $g(x)$.)

3. (a) Suppose $B/C$ is a ring extension and $B \setminus C$ is closed under multiplication. Prove that $C$ is integrally closed in $B$.

(b) Suppose $B/A$ is a ring extension and $C$ is the integral closure of $A$ in $B$. Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

(Hint: Suppose $b \in B$ is integral over $C$, and let $n \in \mathbb{Z}^+$ be the smallest number such that $b^n + c_{n-1}b^{n-1} + \cdots + c_0 = 0$ for some $c_i \in C$.

And show, if $b \in C$, then $n-1$ also satisfies the above property.

(b) Show $B[x] \setminus C[x]$ is closed under multiplication.)

4. (a) Suppose $A$ is a ring and $G$ is a finite subgroup of $\text{Aut}(A)$.

Let $A^G := \{ a \in A \mid \forall \sigma \in G, \sigma(a) = a \}$. Prove that $A/A^G$ is an integral extension.

(b) For $\mathfrak{p} \in \text{Spec}(A^G)$, prove that $G \cap (\mathfrak{p}^{-1})_{\mathfrak{p}}$ transitivity
\[ f: \mathbb{A}^g \rightarrow A. \]

5. Suppose \( k/\mathbb{Q} \) is a finite Galois extension. Let \( \mathcal{O}_k \) be the integral closure of \( \mathbb{Z} \) in \( k \). Suppose \( \mathcal{O}_k \) is a finitely generated ring. Prove that

(a) \( \mathcal{O}_k \) is integrally closed.

(b) \( \dim \mathcal{O}_k = 1 \); and so \( \text{Spec} \mathcal{O}_k = \emptyset \cup \text{Max} \mathcal{O}_k \).

(c) For any prime number \( p \), \( \emptyset \neq \mathfrak{p} \in \text{Spec} (\mathcal{O}_k) \mid \mathfrak{p} \mid \mathcal{O}_k \) is a non-empty finite set, and \( \text{Gal}(k/\mathbb{Q}) \) acts transitively on this set.

(d) \( \forall \mathfrak{a} \neq 0 \neq \mathcal{O}_k, \exists ! \) (up to permutation) primary ideals \( \mathfrak{p}_1, \ldots, \mathfrak{p}_n \) s.t.

\[ \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_n. \]

(e) If \( \mathfrak{p} \) is a non-zero primary ideal, then \( \sqrt{\mathfrak{p}} = \bigcup \mathfrak{m} \in \text{Max} \mathcal{O}_k \) and \( \exists n \in \mathbb{Z}^+, \mathfrak{m}^n \subseteq \mathfrak{p} \).

(f) \( \forall \mathfrak{m} \in \text{Max} (\mathcal{O}_k), \mathcal{O}_k/\mathfrak{m} \) is a finite field.

(g) \( \forall \mathfrak{a} \subset \mathcal{O}_k, \text{ then } N_{\mathcal{O}_k}(\mathfrak{a}) := |\mathcal{O}_k/\mathfrak{a}| < \infty. \)