1. Let $k$ be a finite extension of $\mathbb{Q}$, and $\mathcal{O}_k$ be the integral closure of $\mathbb{Z}$ in $k$. In class we have proved that $\mathcal{O}_k \cong \mathbb{Z}^d$ as an abelian group where $d = [k: \mathbb{Q}]$. Suppose $\mathcal{O}_k = \mathbb{Z} \alpha_1 \oplus \ldots \oplus \mathbb{Z} \alpha_d$, and \[ \text{Hom}_{\mathbb{Q}}(k, \overline{\mathbb{Q}}) = \langle \alpha_1, \ldots, \alpha_d \rangle \] where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. For $\alpha \in k$, let $N_{k/\mathbb{Q}}(\alpha) := \alpha \cdot \alpha \cdots \alpha^d$. 

(a) Prove that $D_k := \det \left[ \sigma_i(\alpha_j) \right] \in \mathbb{Z}$. (It is called the discriminant of $k$.)

(b) Prove that for any $\alpha \in \mathcal{O}_k$, $|N_{k/\mathbb{Q}}(\alpha)| = [\mathcal{O}_k : \alpha \mathcal{O}_k]$. 

(This justifies $N_{k/\mathbb{Q}}(\alpha) := |\mathcal{O}_k / \alpha \mathcal{O}_k|$ for $\alpha \in \mathcal{O}_k$.)

2. Suppose $A$ is a valuation ring of a field $F$, and $A \subseteq A' \subseteq F$ is a subring. Suppose $\text{Max} A = \frac{1}{2} + \mathfrak{P}$ and $\text{Max} A' = \frac{1}{2} + \mathfrak{P}'$. Prove that

(a) $\mathfrak{P}' \subseteq \mathfrak{P}$

(b) $\mathfrak{P}' \in \text{Spec}(A)$ and $A' = A_{\mathfrak{P}'}$

(c) $A_{\mathfrak{P}'}$ is a valuation ring of $A'/\mathfrak{P}'$. 

3. (a) Suppose $\Gamma$ is a totally ordered abelian group, and $F$ is a field. A valuation of $F$ is a function $v : F \to \Gamma \cup \{\infty\}$ with the following properties:

- $v(\alpha) = \infty \iff \alpha = 0$
- $\forall \gamma \in \Gamma$, $\gamma < \infty$, $\gamma + \infty = \infty$, $\infty + \infty = \infty$
- $v(\alpha \beta) = v(\alpha) + v(\beta)$ \quad $\forall \alpha, \beta \in F$
- $v(\alpha + \beta) \geq \min \{v(\alpha), v(\beta)\}$ \quad $\forall \alpha, \beta \in F$

Let $O_v := \{\alpha \in F \mid v(\alpha) \geq 0\}$ and $\text{Max}_v := \{\alpha \in F \mid v(\alpha) > 0\}$.

Prove that $O_v$ is a valuation ring of the field $F$, and $\text{Max}(O_v) = \text{Max}_v$.

(b) Let $A$ be a valuation ring of a field $F$. Let $\Gamma := F^\times / A^\times$.

We say $\alpha A^\times \geq \beta A^\times$ if $\alpha^{-1} \beta \in A$. Prove that $\Gamma$ is a totally ordered abelian group, $v : F \to \Gamma \cup \{\infty\}$,

$v(\alpha) = \begin{cases} \alpha A^\times & \text{if } \alpha \in F^\times \\ \infty & \text{if } \alpha = 0 \in F \end{cases}$

is a valuation of $F$, and $O_v = A$. 

4. (a) Suppose $A$ is a local Noetherian ring with maximal ideal $\mathfrak{m}$, and $M$ is a finitely generated $A$-module. Prove that $M$ is flat $\iff M$ is free.

(b) Suppose $A$ is a Noetherian ring, and $M$ is a f.g. $A$-mod. Prove that the following are equivalent:

(b-1) $M$ is flat.

(b-2) $\forall \mathfrak{p} \in \text{Spec } A$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-mod.

(b-3) $\forall \mathfrak{m} \in \text{Max } A$, $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-mod.

(Hint. (a) $\iff$ Suppose $\overline{x_1, \ldots, x_n}$ is an $A_{\mathfrak{m}}$-basis of $M_{\mathfrak{m}}$. Using Nakayama's lemma show $M=\langle x_1, \ldots, x_n \rangle$. Consider the S.E.S.

$$0 \rightarrow K \rightarrow \mathbb{A}^n \rightarrow M \rightarrow 0$$

Use Math200b, HW 6, problem 4 and deduce

$$0 \rightarrow K \otimes_A A_{\mathfrak{m}} \rightarrow \mathbb{A}^n \otimes_A A_{\mathfrak{m}} \rightarrow M \otimes_A A_{\mathfrak{m}} \rightarrow 0$$

is a S.E.S. Conclude that $K \otimes_A A_{\mathfrak{m}} \cong K_{\mathfrak{m}}$. Using Nakayama's lemma deduce $K=0.$)