Homework 6

Monday, May 28, 2018 9:28 AM

Suppose D is a Noetherian integral domain and, for any $\mathfrak{m} \in \operatorname{Max} D$, $D_{\mathfrak{m}}$ is a PID. Let F be the field of fractions of D. A D-submodule M of F is called a fractional ideal if

$$(D:M) := \{\alpha \in F \mid \alpha M \subseteq D\} \neq 0.$$

(a) Prove that $(D:M)\,M = D$ if $M \neq 0$.

(Hint. $((D:M)M)_{\mathfrak{m}} = (D_{\mathfrak{m}} : M_{\mathfrak{m}})\, M_{\mathfrak{m}}$.)

(b) Prove that M is a finitely generated projective D-mod.

(Hint. $\exists\, \alpha_i \in (D:M)$, $m_i \in M$ s.t. $\alpha_1 m_1 + \cdots + \alpha_n m_n = 1$.

$$0 \to \ker\theta \to D^n \underset{\psi}{\overset{\theta}{\rightleftarrows}} M \to 0$$

$\theta(a_1, \ldots, a_n) := a_1 m_1 + \cdots + a_n m_n$

$\psi(m) := (\alpha_1 m, \alpha_2 m, \ldots, \alpha_n m)$.)

(c) Prove that, for any $\mathfrak{p} \in \operatorname{Spec} D$, $M_{\mathfrak{p}} \simeq D_{\mathfrak{p}}$; and so $[M] \in \operatorname{Pic}(D)$.

(d) Suppose $[N] \in \operatorname{Pic}(D)$; that means N is a f.g. projective mod. s.t., for some $\mathfrak{m} \in \operatorname{Max} D$, $N_{\mathfrak{m}} \simeq D_{\mathfrak{m}}$. Prove that

$N \otimes_D F \longrightarrow F$, $n \otimes 1 \longmapsto n$ is an F-mod isomorphism.

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(Hint. $N\otimes_D F \simeq N\otimes_D D_{\mathfrak{p}} \otimes_{D_{\mathfrak{p}}} F \simeq N_{\mathfrak{p}} \otimes_{D_{\mathfrak{p}}} F \simeq D_{\mathfrak{p}} \otimes_{D_{\mathfrak{p}}} F \simeq F$.

Let $\theta: N\otimes_D F \to F$, $\theta(n\otimes 1) = n$; θ is a well-defined F-linear map and $\operatorname{Im}\theta \neq 0$;)

(e) Suppose M_1 and M_2 are two fractional ideals. Prove that

$$M_1 \simeq M_2 \text{ as } D\text{-modules} \iff \exists \alpha\in F^\times,\ \alpha M_1 = M_2 .$$

(Hint. ($\Rightarrow$)

$$\begin{tikzcd}
m \arrow[r, mapsto] & m\otimes 1 \arrow[r, mapsto] & m\otimes 1 \arrow[r, mapsto] & m \\
M_1 \arrow[r, "\sim"] \arrow[d, "\wr", "\Phi"'] & M_1\otimes_D D \arrow[r, hook] \arrow[d, "\wr"] & M_1\otimes_D F \arrow[r, "\sim"] \arrow[d, "\wr"] & F \arrow[d, "\wr"', "\tilde{\Phi}"] \\
M_2 \arrow[r, "\sim"] & M_2\otimes_D D \arrow[r, hook] & M_2\otimes_D F \arrow[r, "\sim"] & F \\
\Phi(m) \arrow[r, mapsto] & \Phi(m)\otimes 1 \arrow[r, mapsto] & \Phi(m)\otimes 1 \arrow[r, mapsto] & \Phi(m)
\end{tikzcd}$$

.)

(f) Let $\operatorname{Frac}(D) := \{ M\subseteq F \mid M: \text{fractional ideal}\}$;

(f-1) For $M_1, M_2 \in \operatorname{Frac}(D)$, $M_1M_2 \in \operatorname{Frac}(D)$.

Show that $(\operatorname{Frac}(D), \cdot)$ is a group with neutral element D.

(f-2) Let $\operatorname{Prin}(D) := \{\alpha D \mid \alpha\in F\}$. Prove that $\operatorname{Prin}(D)$

is a subgroup of $\operatorname{Frac}(D)$.

(f-3) For $M_1, M_2 \in \operatorname{Frac}(D)$, Prove that $M_1\otimes_D M_2 \to M_1M_2$, $m_1\otimes m_2 \mapsto m_1m_2$

is an isomorphism.

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(g) Let $\theta: \mathrm{Frac}(D) \longrightarrow \mathrm{Pic}(D)$, $\theta(M) := [M]$; prove that θ is a well-defined, onto group homomorphism, and $\ker(\theta) = \mathrm{Prin}(D)$. Deduce that $Cl(D) \simeq \mathrm{Pic}(D)$, where $Cl(D) := \mathrm{Frac}(D)/_{\mathrm{Prin}(D)}$ is the class group of D.