1. Let $\mathcal{O}_k$ be the ring of integers of a number field $k$. Prove that $\mathcal{O}_k$ is a UFD if and only if $\mathcal{O}_k$ is a PID.

2. Suppose $k$ is an algebraically closed field. For $f_1, \ldots, f_{n-1}$ in $k[x_1, \ldots, x_n]$, let $X(f_1, \ldots, f_{n-1}) = \{ p \in k^n \mid f_1(p) = \ldots = f_{n-1}(p) = 0 \}$. Using Krull's height theorem and $\dim k[x_1, \ldots, x_n] = n$, prove that either $X(f_1, \ldots, f_{n-1}) = \emptyset$ or $|X(f_1, \ldots, f_{n-1})| = \infty$.

3. Suppose $\mathcal{O}$ is an integral domain and $A$ is a finitely generated $\mathcal{O}$-algebra. Let $i: \mathcal{O} \to A$ and $i^*: \text{spec } A \to \text{spec } \mathcal{O}$.

For $\mathfrak{p} \in \text{spec } \mathcal{O}$, let $k(\mathfrak{p}) = \text{the field of fractions of } \mathcal{O}_{i^*(\mathfrak{p})}$.

Prove that $\exists \alpha \in \mathcal{O}$ s.t. $\alpha \notin \mathfrak{p} \Rightarrow \dim A \otimes_{\mathcal{O}} k(\mathfrak{p}) = \dim A \otimes_{\mathcal{O}} k(\mathfrak{p})$.

(Hint: $\dim k[x_1, \ldots, x_n] = n$. Use Noether normalization for $A \otimes_{\mathcal{O}} k(\mathfrak{p})$ to find $\alpha \in \mathcal{O} \setminus \mathfrak{p}$ and $x_1, \ldots, x_n \in A$ s.t. $x_i$'s are alg. indep. over $k(\mathfrak{p})$, and $A[\frac{1}{\alpha}]$ is integral over $\mathcal{O}[\frac{1}{\alpha}]][x_1, \ldots, x_n]$. Deduce, if $\alpha \notin \mathfrak{p}$, then $A \otimes_{\mathcal{O}} k(\mathfrak{p})$ is integral over $k(\mathfrak{p})[x_1, \ldots, x_n]$.}
4. Suppose \( A \) is a Noetherian ring and \( \mathfrak{a} \not\subseteq A \). Let

\[ \mathfrak{b}_0 := \bigcap_{i=1}^{\infty} \mathfrak{a}^i. \]

Prove that \( \mathfrak{a} \mathfrak{b}_0 = \mathfrak{b}_0 \).

[Hint. Suppose \( \mathfrak{a} \mathfrak{b}_0 \neq \mathfrak{b}_0 \), and let \( \bigcap_{j=1}^{n} \mathfrak{q}_j \) be a reduced primary decomposition of \( \mathfrak{a} \mathfrak{b}_0 \). So \( \exists j, \mathfrak{b}_0 \not\subseteq \mathfrak{q}_j \).

Suppose \( x \in \mathfrak{b}_0 \setminus \mathfrak{q}_j \). Then \( \mathfrak{a} \subseteq (\mathfrak{q}_j : x) \subseteq \mathfrak{p}_j \). \( \Rightarrow \)

\[ \mathfrak{b}_0 \subseteq \mathfrak{a}^m \subseteq \mathfrak{p}_j^m \subseteq \mathfrak{q}_j \] which is a contradiction.]

5. Suppose \( A \) is a Noetherian local ring and \( \text{Max} \ A = \mathfrak{m}_1 + \mathfrak{m}_2 \).

Prove \( \bigcap_{n=1}^{\infty} \mathfrak{m}_n = 0 \).