Lecture 03: Contraction map and localization

Wednesday, April 4, 2018 10:43 AM

In the previous lecture we stated the following:

<u>Proposition</u>. Suppose S is a multiplicatively closed subset of A, and $0\notin S$. Let $f: A \longrightarrow S^{-1}A$, $f(a) := \frac{a}{1}$. Then f^* induces a bijection from $\mathrm{Spec}(S^{-1}A)$ to $\{\mathfrak{p}\in \mathrm{Spec}(A) \mid S\cap \mathfrak{p}=\varnothing\}$.

<u>Pf</u>. <u>Claim 1</u>. Any ideal $\tilde{\mathfrak{a}}$ of $S^{-1}A$ is of the form $S^{-1}\mathfrak{a}$ where $\mathfrak{a} := \tilde{\mathfrak{a}}^c$.

<u>Pf</u>. By definition $\mathfrak{a} = \{a\in A \mid \frac{a}{1}\in \tilde{\mathfrak{a}}\}$. Clearly $S^{-1}\mathfrak{a} \subseteq \tilde{\mathfrak{a}}$.

Now suppose $\frac{a}{s}\in\tilde{\mathfrak{a}}$. Then $\frac{s}{1}\cdot\frac{a}{s}=\frac{a}{1}\in\tilde{\mathfrak{a}}$; and so $a\in\mathfrak{a}$, which implies $\frac{a}{s}\in S^{-1}\mathfrak{a}$. (here $s\in S$.).

<u>Claim 2</u>. If $\tilde{\mathfrak{p}}\in \mathrm{Spec}(S^{-1}A)$, then $f^*(\tilde{\mathfrak{p}})\cap S=\varnothing$.

<u>Pf</u>. By Claim 1, $\tilde{\mathfrak{p}} = S^{-1}f^*(\tilde{\mathfrak{p}})$. So $S^{-1}f^*(\tilde{\mathfrak{p}})$ is a proper ideal. Therefore $S\cap f^*(\tilde{\mathfrak{p}})=\varnothing$.

<u>Claim 3</u>. $f^*(\tilde{\mathfrak{p}}_1)=f^*(\tilde{\mathfrak{p}}_2) \Rightarrow \tilde{\mathfrak{p}}_1=\tilde{\mathfrak{p}}_2$

<u>Pf</u> By Claim 1, $\tilde{\mathfrak{p}}_1 = S^{-1}f^*(\tilde{\mathfrak{p}}_1)=S^{-1}f^*(\tilde{\mathfrak{p}}_2)=\tilde{\mathfrak{p}}_2$

<u>Claim 4</u>. Suppose $\mathfrak{p}\in\mathrm{Spec}(A)$ and $\mathfrak{p}\cap S=\varnothing$. Then $S^{-1}\mathfrak{p}\in\mathrm{Spec}(S^{-1}A)$.

Pf. Let $\overline{S} := \pi(S)$ where $\pi: A \to A/\mathfrak{p}$ is the natural quotient map. Since $S \cap \mathfrak{p} = \emptyset$, $\bar{0} \notin \pi(S)$. So

$\pi(S)^{-1}(A/\mathfrak{p})$ can be embedded into the field of fractions

$Q(A/\mathfrak{p})$ of the integral domain $A/\mathfrak{p}$. On the other hand,

$S^{-1}A/S^{-1}\mathfrak{p} \simeq \pi(S)^{-1}(A/\mathfrak{p})$ (why?). Hence $S^{-1}A/S^{-1}\mathfrak{p}$ is

an integral domain; and so $S^{-1}\mathfrak{p} \in \operatorname{Spec}(S^{-1}A)$.

Claim 5. Suppose $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{p} \cap S = \emptyset$. Then $f^*(S^{-1}\mathfrak{p}) = \mathfrak{p}$.

Pf. Clearly $\mathfrak{p} \subseteq f^*(S^{-1}\mathfrak{p})$. Now suppose $\frac{a}{1} \in S^{-1}\mathfrak{p}$. Then

$\exists\, s \in S$ s.t. $sa \in \mathfrak{p} \Rightarrow$ either $s \in \mathfrak{p}$ or $a \in \mathfrak{p}$. As $S \cap \mathfrak{p} = \emptyset$

we deduce that $a \in \mathfrak{p}$; and claim follows.

By claim 2, f^* induces a function from $\operatorname{Spec}(S^{-1}A)$ to

$\{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\}$; by claim 3, f^* is injective;

by Claim 4 and 5, we get the image of f^*. ■

Lecture 03: Spec of localization

Friday, April 6, 2018 8:07 AM

<u>Corollary</u>. Suppose $\mathfrak{p} \in \mathrm{Spec}(A)$, and $f: A \to A_{\mathfrak{p}}$, $f(a) = \frac{a}{1}$

(recall $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$ is multiplicatively closed, and $A_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1} A$.).

Then $f^*: \mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$ induces a bijection from

$\mathrm{Spec}(A_{\mathfrak{p}})$ to $\{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} \subseteq \mathfrak{p}\}$

<u>Corollary</u>. Suppose $f: A \to B$ is a ring homomorphism, and

$\mathfrak{p} \in \mathrm{Spec}(A)$. Let $k(\mathfrak{p})$ be the residue field at $\mathfrak{p}$; that means

$k(\mathfrak{p}) := A_{\mathfrak{p}} / \mathfrak{p} A_{\mathfrak{p}}$. Then $(f^*)^{-1}(\mathfrak{p})$ is in bijection with $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$.

<u>Pf</u>. <u>Claim 1</u>. $A_{\mathfrak{p}} \otimes_A B \simeq f(S_{\mathfrak{p}})^{-1} B$

$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$.

<u>Pf</u>. For an A-mod M, we have proved $S^{-1}A \otimes_A M \simeq S^{-1}M$;

$f: A \to B$ makes B into an A-mod via $a \cdot b := f(a)\, b$

scalar multiplication. And so as A-modules we get that

$A_{\mathfrak{p}} \otimes_A B \xrightarrow[\sim]{\theta} f(S_{\mathfrak{p}})^{-1} B$, $\frac{a}{s} \otimes b \longmapsto \frac{f(a)\, b}{f(s)}$. Notice that

$$\left(\frac{a_1}{s_1} \otimes b_1\right)\left(\frac{a_2}{s_2} \otimes b_2\right) = \frac{a_1 a_2}{s_1 s_2} \otimes b_1 b_2 \longmapsto \frac{f(a_1 a_2)\, b_1 b_2}{f(s_1 s_2)}$$

$$\frac{f(a_1)\, b_1}{f(s_1)} \cdot \frac{f(a_2)\, b_2}{f(s_2)} = \frac{f(a_1 a_2)\, b_1 b_2}{f(s_1 s_2)}$$

. So θ is a ring homo. $\square$

Lecture 03: Fiber over a prime

Friday, April 6, 2018 8:03 AM

Claim 2. $B \otimes_A k(\mathfrak{p}) \simeq f(S_\mathfrak{p})^{-1}B \big/ f(S_\mathfrak{p})^{-1}\langle f(\mathfrak{p})\rangle$.

Pf. We have proved that if $g: R \to R'$ is a ring homo.,

then, for any $I \triangleleft R$, $R' \otimes_R R/I \simeq R'/g(I)R' = R'/I^e$.

$$B \otimes_A k_\mathfrak{p} \simeq B \otimes_A \left(A_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}\right) \simeq (B \otimes_A A_\mathfrak{p}) \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$$

$$\simeq f(S_\mathfrak{p})^{-1}B \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \simeq f(S_\mathfrak{p})^{-1}B \big/ f(S_\mathfrak{p})^{-1} \cdot \langle f(\mathfrak{p})\rangle.$$

(We will continue next time)