Lecture 05: Primary ideals

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<u>Def.</u> $\mathfrak{q} \underset{\neq}{\trianglelefteq} A$ is called a <u>primary ideal</u> if $xy \in \mathfrak{q} \Rightarrow$ either $x \in \mathfrak{q}$ or $\exists n \in \mathbb{Z}^+$, $y^n \in \mathfrak{q}$.

<u>Lemma</u>. $\mathfrak{q}$ is primary if and only if any zero-divisor of $A/\mathfrak{q}$ is nilpotent.

<u>Pf.</u> ($\Rightarrow$) Suppose $\bar{x} \cdot \bar{y} = 0$ in $A/\mathfrak{q}$. Then either $\bar{x} = 0$ or $\bar{y}^n = 0$. And claim follows.

($\Leftarrow$) Suppose $x \cdot y \in \mathfrak{q}$. Then $\bar{x} \cdot \bar{y} = 0$. So either $\bar{x} = 0$ or $\bar{y}$ is a zero-divisor. Hence either $x \in \mathfrak{q}$ or $\bar{y}$ is a zero-div. in $A/\mathfrak{q}$. In the latter case $\bar{y}$ is nilpotent; and so $y^n \in \mathfrak{q}$ for some $n \in \mathbb{Z}^+$. ■

<u>Lemma</u>. Suppose $\mathfrak{q}$ is primary. Then $\sqrt{\mathfrak{q}}$ is prime; and so it is the smallest prime divisor of $\mathfrak{q}$.

<u>Pf.</u> Suppose $x \cdot y \in \sqrt{\mathfrak{q}}$ and $x, y \notin \sqrt{\mathfrak{q}}$. Then $\exists n \in \mathbb{Z}^+$ s.t. $x^n \cdot y^n \in \mathfrak{q}$; and $x^n \notin \mathfrak{q}$ and $y^n \notin \sqrt{\mathfrak{q}}$ which contradicts the fact that $\mathfrak{q}$ is primary.; on the other hand, $\sqrt{\mathfrak{q}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{q})} \mathfrak{p}$. ■

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<u>Def.</u> We say $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\mathfrak{q}$ is primary and $\sqrt{\mathfrak{q}} = \mathfrak{p}$.

<u>Lemma.</u> Suppose $\mathfrak{m} \in \mathrm{Max}(A)$ and, for $\mathfrak{q} \lhd A$, $\sqrt{\mathfrak{q}} = \mathfrak{m}$. Then $\mathfrak{q}$ is $\mathfrak{m}$-primary.

<u>Pf.</u> <u>Claim</u> $V(\mathfrak{q}) = \{\mathfrak{m}\}$.

<u>Pf.</u> By the previous lemma $V(\mathfrak{q})$ has a (unique) smallest element which is $\mathfrak{m}$ in this case. Since $\mathfrak{m} \in \mathrm{Max}(A)$, we deduce $V(\mathfrak{q}) = \{\mathfrak{m}\}$. ■

- Suppose $xy \in \mathfrak{q}$. Consider $(\mathfrak{q} : x) := \{a \in A \mid ax \in \mathfrak{q}\}$.

Then $(\mathfrak{q} : x) \lhd A$, $(\mathfrak{q} : x) \mid \mathfrak{q}$, $y \in (\mathfrak{q} : x)$.

Since $(\mathfrak{q} : x) \mid \mathfrak{q}$, $V(\mathfrak{q} : x) \subseteq V(\mathfrak{q}) = \{\mathfrak{m}\}$.

So either $V(\mathfrak{q} : x) = \emptyset$ or $V(\mathfrak{q} : x) = \{\mathfrak{m}\}$.

<u>Case 1.</u> $V(\mathfrak{q} : x) = \emptyset \Rightarrow (\mathfrak{q} : x) = A \Rightarrow x \in \mathfrak{q}$

<u>Case 2.</u> $V(\mathfrak{q} : x) = \mathfrak{m} \Rightarrow y \in (\mathfrak{q} : x) \subseteq \sqrt{(\mathfrak{q} : x)} = \mathfrak{m}$. ■

<u>Corollary.</u> Suppose $\mathfrak{m} \in \mathrm{Max}(A)$ and $k \in \mathbb{Z}^+$. Then $\mathfrak{m}^k$ is $\mathfrak{m}$-primary.

<u>Pf.</u> $\mathfrak{p} \in V(\mathfrak{m}^k) \Leftrightarrow \mathfrak{m}^k \subseteq \mathfrak{p} \Leftrightarrow \mathfrak{m} \subseteq \mathfrak{p} \Leftrightarrow \mathfrak{m} = \mathfrak{p}$.

And so $\sqrt{\mathfrak{m}^k} = \mathfrak{m}$; and claim follows from the previous lemma. ■

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<u>Corollary</u>. In a PID, a primary is either 0 or $\langle p^k \rangle$ where p is irreducible.

<u>Pf.</u> Let $\mathfrak{q}$ be a primary ideal. Then $\sqrt{\mathfrak{q}} \in \operatorname{Spec}(A) = \{0\} \cup \operatorname{Max}(A)$.

If $\sqrt{\mathfrak{q}} = 0$, then $\mathfrak{q} = 0$. If not, $\sqrt{\mathfrak{q}} = \langle p \rangle$ where p is irreducible.

If $\mathfrak{q} = \langle a \rangle$, then p is the only irredu. factor of a; and so $\mathfrak{q} = \langle p^k \rangle$.

If $\mathfrak{q} = \langle p^k \rangle$, then $\sqrt{\mathfrak{q}} = \langle p \rangle \in \operatorname{Max}(A)$; and so $\mathfrak{q}$ is primary. ∎

<u>Proposition</u>. Suppose $\mathfrak{q}$ is $\mathfrak{p}$-primary. Then

(1) $(\mathfrak{q} : x) = A$ if and only if $x \in \mathfrak{q}$.

(2) $(\mathfrak{q} : x) = \mathfrak{q}$ if $x \notin \mathfrak{p}$.

(3) $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary if $x \notin \mathfrak{q}$.

<u>Pf.</u> (1) is clear.

(2) Suppose $y \in (\mathfrak{q} : x)$. Then $xy \in \mathfrak{q}$. Since $x \notin \sqrt{\mathfrak{q}}$, $\mathfrak{q}$ is primary, and $xy \in \mathfrak{q}$, we have $y \in \mathfrak{q}$.

(3). $y \in \sqrt{(\mathfrak{q} : x)} \Rightarrow \left.\begin{array}{l} y^n x \in \mathfrak{q} \\ x \notin \mathfrak{q} \end{array}\right\} \Rightarrow y^n \in \sqrt{\mathfrak{q}} \Rightarrow y \in \sqrt{\mathfrak{q}} = \mathfrak{p}$

Clearly $\sqrt{\mathfrak{q}} \subseteq \sqrt{(\mathfrak{q} : x)}$; and so $\mathfrak{p} = \sqrt{(\mathfrak{q} : x)}$.

- Suppose $yz \in (\mathfrak{q} : x)$ and $z \notin \sqrt{(\mathfrak{q} : x)} = \mathfrak{p}$. And so $xyz \in \mathfrak{q}$.

$\left.\begin{array}{l} xyz \in \mathfrak{q} \\ z \notin \sqrt{\mathfrak{q}} \end{array}\right\} \Rightarrow xy \in \mathfrak{q} \Rightarrow y \in (\mathfrak{q} : x)$.

∎

Lecture 05: Primary decomposition

Monday, April 9, 2018 7:45 AM

Def. Suppose $\mathfrak{a} \lhd A$. A primary decomposition of $\mathfrak{a}$ is

$$\mathfrak{a} = \bigcap_{i=1}^{m} \mathfrak{q}_i$$ where $\mathfrak{q}_i$'s are $\mathfrak{p}_i$-primary.

- We say $\mathfrak{a}$ is decomposable if it has a primary decomposition.
- We say it is a reduced primary decomposition if

$$\forall i, \ \mathfrak{q}_i \not\supseteq \bigcap_{\substack{j=1 \\ j\neq i}}^{m} \mathfrak{q}_j \quad \text{and} \quad \forall i \neq j, \ \mathfrak{p}_i \neq \mathfrak{p}_j.$$

Lemma. (1) If $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then $\mathfrak{q} \cap \mathfrak{q}'$ is $\mathfrak{p}$-primary.

(2) A decomposable ideal has a reduced primary decomp.

We will continue next time.