Lecture 07: Primary ideals of ring of fractions

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At the end of the previous lecture we were proving the following propo.

Proposition. Let $f: A \to S^{-1}A$, $f(a) = \frac{a}{1}$.

(1) Suppose $\mathfrak{q}$ is $\mathfrak{p}$-primary, and $\mathfrak{p} \cap S \neq \emptyset$. Then $S^{-1}\mathfrak{q} = S^{-1}A$.

(2) Suppose $\mathfrak{q}$ is $\mathfrak{p}$-primary, and $\mathfrak{p} \cap S = \emptyset$. Then $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary.

(3) Suppose $\tilde{\mathfrak{q}}$ is $\tilde{\mathfrak{p}}$-primary, $\mathfrak{q} = \tilde{\mathfrak{q}}^c$, and $\mathfrak{p} = \tilde{\mathfrak{p}}^c$. Then $\mathfrak{q}$ is $\mathfrak{p}$-primary.

(4) Consider the maps induced by the contraction ane extension maps:

$$\{\mathfrak{q} \trianglelefteq A \mid \mathfrak{q} : \mathfrak{p}\text{-primary}, \mathfrak{p} \cap S = \emptyset\} \underset{c}{\overset{e}{\rightleftarrows}} \{\tilde{\mathfrak{q}} \trianglelefteq S^{-1}A \mid \tilde{\mathfrak{p}}\text{-primary}\}.$$

Then these are inverse of each other.

Pf. We have already proved (1). (2) We start by proving $\sqrt{S^{-1}\mathfrak{q}} = S^{-1}\mathfrak{p}$. Notice that $S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{p}$ and $S^{-1}\mathfrak{p}$ is prime as $\mathfrak{p} \cap S = \emptyset$. Hence $\sqrt{S^{-1}\mathfrak{q}} \subseteq S^{-1}\mathfrak{p}$. On the other hand, $S^{-1}\sqrt{\mathfrak{q}} \subseteq \sqrt{S^{-1}\mathfrak{q}}$; and so $\sqrt{S^{-1}\mathfrak{q}} = S^{-1}\mathfrak{p}$.

$$\left.\begin{array}{l}\frac{x}{s} \cdot \frac{y}{s'} \in S^{-1}\mathfrak{q} \Rightarrow \exists\, s'' \in S,\ s''x\,y \in \mathfrak{q} \\ \frac{x}{s} \notin S^{-1}\mathfrak{p} \Rightarrow x \notin \mathfrak{p}\end{array}\right\} \Rightarrow \left.\begin{array}{l}(s''x)\,y \in \mathfrak{q} \\ s''x \notin \mathfrak{p}\end{array}\right\} \Rightarrow y \in \mathfrak{q} \Rightarrow \frac{y}{s'} \in S^{-1}\mathfrak{q}.$$

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(3) Suppose $\tilde{\mathfrak{q}}$ is $\tilde{\mathfrak{p}}$-primary, $\mathfrak{q} = \tilde{\mathfrak{q}}^c$, $\mathfrak{p} = \tilde{\mathfrak{p}}^c$. Then $\mathfrak{p} \in \operatorname{Spec}(A)$, $\mathfrak{p} \cap S = \varnothing$, and $\tilde{\mathfrak{p}} = S^{-1}\mathfrak{p}$; and $\tilde{\mathfrak{q}} = S^{-1}\mathfrak{q}$.

- $x \in \mathfrak{p} \Leftrightarrow \frac{x}{1} \in \tilde{\mathfrak{p}} \Leftrightarrow \exists n \in \mathbb{Z}^+,\ \frac{x^n}{1} \in \tilde{\mathfrak{q}} \Leftrightarrow \exists n \in \mathbb{Z}^+,\ x^n \in \mathfrak{q}$.

Hence $\sqrt{\mathfrak{q}} = \mathfrak{p}$.

- $\left.\begin{matrix} xy \in \mathfrak{q} \\ x \notin \mathfrak{p} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \frac{xy}{1} \in \tilde{\mathfrak{q}} \\ \frac{x}{1} \notin \tilde{\mathfrak{p}} \end{matrix}\right\} \Rightarrow \frac{y}{1} \in \tilde{\mathfrak{q}} \Rightarrow y \in \mathfrak{q}$.

(4) By (2), we showed $\underline{e}$ is well-defined. By (1), we get that $\underline{c}$ is well-defined; and by (3), we get that $\underline{e}$ is onto.

For any $\tilde{\mathfrak{a}} \lhd S^{-1}A$, we have $(\tilde{\mathfrak{a}}^c)^e = \tilde{\mathfrak{a}}$. For $\mathfrak{a} \lhd A$, let $S(\mathfrak{a}) := (\mathfrak{a}^e)^c$. So it is only remaind to show $S(\mathfrak{q}) = \mathfrak{q}$ if $\mathfrak{q}$ is $\mathfrak{p}$-primary and $\mathfrak{p} \cap S = \varnothing$.

- $x \in S(\mathfrak{q}) \Rightarrow \frac{x}{1} \in S^{-1}\mathfrak{q} \Rightarrow \exists s \in S,\ sx \in \mathfrak{q} \Rightarrow \left\{\begin{matrix} sx \in \mathfrak{q} \\ s \notin \mathfrak{p} \end{matrix}\right\} \Rightarrow x \in \mathfrak{q}$.
- Clearly $S(\mathfrak{a}) \supseteq \mathfrak{a}$. ∎

Notice that $\mathfrak{a} \subseteq S(\mathfrak{a})$; $\mathfrak{p} = S(\mathfrak{p})$ if $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{p} \cap S = \varnothing$; $\mathfrak{q} = S(\mathfrak{q})$ if $\mathfrak{q}$ is $\mathfrak{p}$-primary and $\mathfrak{p} \cap S = \varnothing$.

Lecture 07: Primary decompositions and localization

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What happens to a primary decomposition after a localization?

Lemma. Suppose $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is a reduced primary decomposition, $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, $S \subseteq A$ is a multiplicatively closed subset, $S \cap \mathfrak{p}_j = \emptyset$ for $1 \leq j \leq m$, and $S \cap \mathfrak{p}_j \neq \emptyset$ for $m < j \leq n$. Then

$$S^{-1}\mathfrak{a} = \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i \quad \text{and} \quad S(\mathfrak{a}) = \bigcap_{i=1}^{m} \mathfrak{q}_i .$$

Pf. • $\mathfrak{a} \subseteq \mathfrak{q}_i \Rightarrow S^{-1}\mathfrak{a} \subseteq S^{-1}\mathfrak{q}_i \Rightarrow \left.\begin{array}{l} S^{-1}\mathfrak{a} \subseteq \bigcap_{i=1}^{n} S^{-1}\mathfrak{q}_i \\ S^{-1}\mathfrak{q}_i = S^{-1}A \text{ if } m < i \leq n \end{array}\right\} \Rightarrow S^{-1}\mathfrak{a} \subseteq \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i$. (1)

• $\frac{x}{s} \in \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i \Rightarrow \frac{x}{1} \in \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i \Rightarrow x \in \bigcap_{i=1}^{m} S(\mathfrak{q}_i) = \bigcap_{i=1}^{m} \mathfrak{q}_i$

Let $s_j \in S \cap \mathfrak{p}_j$ for $m < j \leq n$. So $s_{m+1} \cdot \dots \cdot s_n \cdot x \in \bigcap_{i=1}^{n} \mathfrak{q}_i = \mathfrak{a}$. Hence

$$\frac{x}{s} = \frac{s_{m+1} \cdot \dots \cdot s_n \, x}{s_{m+1} \cdot \dots \cdot s_n \, s} \in S^{-1}\mathfrak{a}. \quad (2)$$

(1) & (2) imply $S^{-1}\mathfrak{a} = \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i$.

• $S(\mathfrak{a}) = (S^{-1}\mathfrak{a})^c = \left(\bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i\right)^c = \bigcap_{i=1}^{m} (S^{-1}\mathfrak{q}_i)^c = \bigcap_{i=1}^{m} \mathfrak{q}_i$. ■

Def. $\Sigma \subseteq \operatorname{Ass}(\mathfrak{a})$ is called *isolated* if

$$\forall \mathfrak{p} \in \Sigma, \ \mathfrak{p}' \in \operatorname{Ass}(\mathfrak{a}), \ \mathfrak{p}' \subseteq \mathfrak{p} \Rightarrow \mathfrak{p}' \in \Sigma .$$

Ex. If $\mathfrak{p}$ is a minimal element of $\operatorname{Ass}(\mathfrak{a})$, then $\{\mathfrak{p}\}$ is isolated.

Lecture 07: Second uniqueness theorem

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<u>Theorem</u>. Suppose $\mathfrak{a} \trianglelefteq A$ is decomposable, $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is a reduced primary decomposition, and $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Suppose $\Sigma := \{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_m}\} \subseteq \mathrm{Ass}(\mathfrak{a})$ is isolated; that means

$$\mathfrak{p} \in \Sigma, \ \mathfrak{p}' \in \mathrm{Ass}(\mathfrak{a}), \ \mathfrak{p}' \subseteq \mathfrak{p} \implies \mathfrak{p}' \in \Sigma.$$

Then $\bigcap_{j=1}^{m} \mathfrak{q}_{i_j}$ just depends on Σ (it is independent of the choice of the reduced primary decomposition $\bigcap_{i=1}^{n} \mathfrak{q}_i$).

In particular, if $\mathfrak{p}$ is a minimal prime ideal associated with $\mathfrak{a}$, then the $\mathfrak{p}$-primary factor $\mathfrak{q}$ is unique.

<u>Pf</u>. Let $S_\Sigma := A \setminus \left(\bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}\right)$. Then S_Σ is a multiplicatively closed set; $\forall \mathfrak{p} \in \Sigma$, $S_\Sigma \cap \mathfrak{p} = \emptyset$; $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{a}) \setminus \Sigma$, then $\forall \mathfrak{p}' \in \Sigma$, $\mathfrak{p} \not\subseteq \mathfrak{p}'$. Hence $\mathfrak{p} \not\subseteq \bigcup_{\mathfrak{p}' \in \Sigma} \mathfrak{p}'$.

$\Rightarrow \mathfrak{p} \cap S_\Sigma \neq \emptyset$. Therefore by the previous lemma

$$S_\Sigma(\mathfrak{a}) = \bigcap_{j=1}^{m} \mathfrak{q}_{i_j}.$$

- If $\mathfrak{p}$ is a minimal prime ideal associated with $\mathfrak{a}$, then $\Sigma = \{\mathfrak{p}\}$ is isolated; and the claim follows. ∎