At the end of the previous lecture we were proving the following proposition.

**Proposition.** Let \( f: A \to S^{-1}A \), \( f(a) = \frac{a}{1} \).

1. Suppose \( \mathfrak{q} \) is \( \mathfrak{p} \)-primary, and \( \mathfrak{p} \cap S \neq \emptyset \). Then \( S^{-1}\mathfrak{q} = S^{-1}A \).

2. Suppose \( \mathfrak{q} \) is \( \mathfrak{p} \)-primary, and \( \mathfrak{p} \cap S = \emptyset \). Then \( S^{-1}\mathfrak{q} \) is \( \mathfrak{p}^{-1} \)-primary.

3. Suppose \( \mathfrak{q} \) is \( \mathfrak{p}^{-1} \)-primary, \( \mathfrak{q} = \mathfrak{q}^c \), and \( \mathfrak{p} = \mathfrak{p}^c \). Then \( \mathfrak{q} \) is \( \mathfrak{p} \)-primary.

4. Consider the maps induced by the contraction one extension maps:

\[ \left\{ \begin{array}{c}
\{ \mathfrak{q} \in A | \mathfrak{q} \text{ is } \mathfrak{p} \text{-primary, } \mathfrak{p} \cap S = \emptyset \} \\
\{ \mathfrak{q} \in S^{-1}A | \mathfrak{q} \text{ is } \mathfrak{p}^{-1} \text{-primary} \}
\end{array} \right. \]

Then these are inverse of each other.

**Pf:** We have already proved (1). (2) We start by proving

\[ \sqrt{S^{-1}\mathfrak{q}} = S^{-1}\mathfrak{p} \]. Notice that \( S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{p} \) and \( S^{-1}\mathfrak{p} \) is prime as \( \mathfrak{p} \cap S = \emptyset \). Hence \( \sqrt{S^{-1}\mathfrak{q}} \subseteq S^{-1}\mathfrak{p} \). On the other hand, \( S^{-1}\sqrt{S^{-1}\mathfrak{q}} \subseteq \sqrt{S^{-1}\mathfrak{q}} \), and so \( \sqrt{S^{-1}\mathfrak{q}} = S^{-1}\mathfrak{p} \).

\[ \frac{x}{s} \cdot \frac{y}{s} \in S^{-1}\mathfrak{q} \Rightarrow \exists s' \in S, s'x y \in \mathfrak{q} \Rightarrow s'(s'x) y \in \mathfrak{q} \Rightarrow y \in \mathfrak{q} \]

\[ \frac{x}{s} \notin S^{-1}\mathfrak{p} \Rightarrow x \notin \mathfrak{p} \]

\[ \frac{y}{s'} \in S^{-1}\mathfrak{q} \]
Lecture 07: Primary ideals of ring of fractions

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(3) Suppose \( \mathfrak{q} \) is \( \mathfrak{p} \)-primary, \( \mathfrak{q} = \mathfrak{q}^c, \mathfrak{p}^c = \mathfrak{p}^c \). Then \( \mathfrak{p} \in \text{Spec}(A) \), \( \mathfrak{p} \cap S = \emptyset \), and \( \mathfrak{q} = S^{-1}\mathfrak{p} \); and \( \mathfrak{q} = S^{-1}\mathfrak{q} \).

\[ \begin{align*}
\text{If } & x \in \mathfrak{q} \iff \frac{x}{1} \in \mathfrak{q} \iff \exists n \in \mathbb{Z}^+, \frac{x^n}{1} \in \overline{\mathfrak{q}} \iff \exists n \in \mathbb{Z}^+, x^n \in \mathfrak{q}.
\end{align*} \]

Hence \( \sqrt{\mathfrak{q}} = \mathfrak{p} \).

\[ \begin{align*}
\text{If } & xy \in \mathfrak{q} \implies \frac{xy}{1} \in \mathfrak{q} \implies \frac{y}{1} \in \overline{\mathfrak{q}} \implies y \in \mathfrak{q}.
\end{align*} \]

(4) By (2), we showed \( e \) is well-defined. By (1), we get that \( e \) is well-defined; and by (3), we get that \( e \) is onto.

For any \( \mathfrak{a} \triangleleft S^{-1}A \), we have \( (\mathfrak{a}^c)^e = \mathfrak{a} \). For \( \mathfrak{a} \triangleleft A \), let \( S(\mathfrak{a}) := (\mathfrak{a}^c)^c \). So it is only remain to show \( S(\mathfrak{q}) = \mathfrak{q} \) if \( \mathfrak{q} \)
is \( \mathfrak{p} \)-primary and \( \mathfrak{p} \cap S = \emptyset \).

\[ \begin{align*}
\text{If } & x \in S(\mathfrak{q}) \iff \frac{x}{1} \in S^{-1}\mathfrak{q} \iff \exists s \in S, sx \in \mathfrak{q} \iff s \cdot x \in \mathfrak{q}.
\end{align*} \]

Clearly \( S(\mathfrak{a}) \supseteq \mathfrak{a} \). \( \blacksquare \)

Notice that \( \mathfrak{a} \subseteq S(\mathfrak{a}) \); \( \mathfrak{p} = S(\mathfrak{p}) \) if \( \mathfrak{p} \in \text{Spec}(A) \) and \( \mathfrak{p} \cap S = \emptyset \); \( \mathfrak{q} = S(\mathfrak{q}) \) if \( \mathfrak{q} \) is \( \mathfrak{p} \)-primary and \( \mathfrak{p} \cap S = \emptyset \).
What happens to a primary decomposition after a localization?

**Lemma.** Suppose \( \mathcal{U} = \bigcap_{i=1}^{n} \mathfrak{q}_i \) is a reduced primary decomposition, \( \mathfrak{q}_i \) is \( \mathfrak{p}_i \)-primary, \( S \subseteq \mathcal{A} \) is a multiplicatively closed subset, \( S \mathfrak{p}_j = \emptyset \) for \( 1 \leq j \leq m \), and \( S \mathfrak{p}_j \neq \emptyset \) for \( m < j \leq n \). Then

\[
S^{-1} \mathcal{U} = \bigcap_{i=1}^{m} S^{-1} \mathfrak{q}_i \quad \text{and} \quad S(\mathcal{U}) = \bigcap_{i=1}^{m} \mathfrak{q}_i.
\]

**Proof.** \( \mathcal{U} \subseteq \mathfrak{q}_i \Rightarrow S^{-1} \mathcal{U} \subseteq S^{-1} \mathfrak{q}_i \Rightarrow S^{-1} \mathcal{U} \subseteq \bigcap_{i=1}^{n} S^{-1} \mathfrak{q}_i \Rightarrow S^{-1} \mathcal{U} \subseteq \bigcap_{i=1}^{m} S^{-1} \mathfrak{q}_i \). \[ (1) \]

\[
\frac{\alpha}{s} \in \bigcap_{i=1}^{m} S^{-1} \mathfrak{q}_i \Rightarrow \frac{\alpha}{s} \in \bigcap_{i=1}^{m} S^{-1} \mathfrak{q}_i \Rightarrow \alpha \in \bigcap_{i=1}^{m} S(\mathfrak{q}_i) = \bigcap_{i=1}^{m} \mathfrak{q}_i.
\]

Let \( s_j \in S \mathfrak{p}_j \) for \( m < j \leq n \). So \( s_1 \cdots s_n \alpha \in \bigcap_{i=1}^{n} \mathfrak{q}_i \), hence

\[
\frac{\alpha}{s} = \frac{s_{m+1} \cdots s_n \alpha}{s_{m+1} \cdots s_n s} \in S^{-1} \mathcal{U}. \quad (2)
\]

(1) \& (2) imply \( S^{-1} \mathcal{U} = \bigcap_{i=1}^{m} S^{-1} \mathfrak{q}_i \).

\[
S(\mathcal{U}) = (S^{-1} \mathcal{U})^c = \left( \bigcap_{i=1}^{m} S^{-1} \mathfrak{q}_i \right)^c = \bigcap_{i=1}^{m} (S^{-1} \mathfrak{q}_i)^c = \bigcap_{i=1}^{m} \mathfrak{q}_i.
\]

**Def.** \( \Sigma \subset \text{Ass}(\mathcal{U}) \) is called isolated if

\[
\forall \mathfrak{p}, \mathfrak{q} \in \Sigma, \mathfrak{p} \subsetneq \mathfrak{q} \Rightarrow \mathfrak{p} \cap \mathfrak{q} = \emptyset.
\]

**Ex.** If \( \mathfrak{p} \) is a minimal element of \( \text{Ass}(\mathcal{U}) \), then \( \mathfrak{p} \subsetneq \mathfrak{q} \) is isolated.
**Theorem.** Suppose \( \mathfrak{A} \triangleleft A \) is decomposable, \( \mathfrak{A} = \bigcap_{i=1}^{n} \mathfrak{p}_i \) is a reduced primary decomposition, and \( \mathfrak{p}_i \) is \( \mathfrak{p}_i \)-primary. Suppose

\[ \Sigma := \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_m \} \subseteq \text{Ass}(\mathfrak{A}) \text{ is isolated;} \text{ that means} \]

\[ \mathfrak{p} \in \Sigma, \mathfrak{p}' \in \text{Ass}(\mathfrak{A}), \mathfrak{p}' \subseteq \mathfrak{p} \implies \mathfrak{p}' \in \Sigma. \]

Then \( \bigcap_{i=1}^{m} \mathfrak{p}_i \) just depends on \( \Sigma \) (it is independent of the choice of the reduced primary decomposition \( \bigcap_{i=1}^{n} \mathfrak{p}_i \)).

In particular, if \( \mathfrak{p} \) is a minimal prime ideal associated with \( \mathfrak{A} \), then the \( \mathfrak{p} \)-primary factor \( \mathfrak{q}_i \) is unique.

---

**Proof.** Let \( \Sigma_i := A \setminus (\bigcup_{\mathfrak{q} \in \Sigma_i} \mathfrak{q}) \). Then \( \Sigma_i \) is a multiplicatively closed set; \( \forall \mathfrak{p} \in \Sigma \), \( \Sigma_i \mathfrak{p} = \emptyset \); \( \forall \mathfrak{p} \in \text{Ass}(\mathfrak{A}) \setminus \Sigma \),

then \( \forall \mathfrak{p} \in \Sigma, \mathfrak{p} \neq \mathfrak{p}' \). Hence \( \mathfrak{p} \neq \bigcup_{\mathfrak{q} \in \Sigma} \mathfrak{q} \).

\[ \mathfrak{p} \cap \Sigma_i \neq \emptyset . \text{ Therefore by the previous lemma} \]

\[ S_{\Sigma_i}(\mathfrak{A}) = \bigcap_{j=1}^{m} \mathfrak{p}_j \]

is isolated; and the claim follows. \( \blacksquare \)