In the previous lecture we proved the 2nd uniqueness theorem which implies:

If \( R \) is decomposable, for any minimal element \( \wp \) of \( \text{Ass}(R) \) \( \wp \)-factors of reduced primary decompositions of \( R \) are the same.

We will see an application of this for Krull dimension 1 integral domains.

**Def.** \( \dim A := \sup_{\wp \in \mathbb{Z}^+} |\wp| \text{Spec}(A), \wp \in \mathbb{Z}^+ \) \( \mathbb{Z} \).

**Ex.** \( k: \text{field} \implies \dim k = 0 \).

**Ex.** Suppose \( A \) is an integral domain. Then \( \dim A = 1 \) if and only if \( 0 \notin \text{Max}(A) \) and \( \text{Spec}(A) = \text{Spec}(\mathbb{Z}) \cup \text{Max}(A) \); in particular \( \dim(A) = 1 \) if \( A \) is a PID and not a field.

**If.** \( (\Rightarrow) \) If \( 0 \notin \text{Max}(A) \), then \( \dim A = 0 \); that is a contradiction.

Suppose to the contrary \( \exists \wp \in \text{Spec}(A) \setminus (\mathbb{Z} \cup \text{Max}(A)) \). Then \( \exists \wp \in \text{Max}(A) \) s.t. \( \wp \nmid \wp \implies 0 \nmid \wp \implies \dim A \geq 2 \); this is a contradiction. \( (\Leftarrow) \) Since \( 0 \notin \text{Max}(A) \), for \( \wp \in \text{Max}(A) \) we get \( 0 \nmid \wp \). And so \( \dim A \geq 1 \). If \( \text{Spec}(A) \setminus (\mathbb{Z} \cup \text{Max}(A)) \), which is a contradiction.
Lecture 08: Dimension one integral domains

Monday, April 16, 2018 12:25 AM

We have seen that, if \( A \) is a PID, then \( \text{Spec}(A) = \text{nil}(A) \cup \text{Max}(A) \).

And claim follows.

**Proposition.** Suppose \( D \) is an integral domain and \( \dim D = 1 \).

Suppose \( \alpha \triangleleft D \) is decomposable. Then \( \alpha \) has a unique reduced primary decomposition.

**Pf.** If \( \alpha = 0 \), then \( \alpha \) is prime; and so by the 1st uniqueness theorem we are done.

- If \( \alpha \neq 0 \), then \( \text{Ass}(\alpha) \subseteq \text{Max}(A) \); and so any element of \( \text{Ass}(\alpha) \) is minimal in \( \text{Ass}(\alpha) \). Hence by the 2nd uniqueness theorem claim follows. ■

**Corollary.** Suppose \( D \) is an integral domain and \( \dim D = 1 \).

Suppose \( \neq \alpha \triangleleft D \) is decomposable. Then there are unique primary ideals (up to permutation) \( q_1, \ldots, q_n \) s.t.

\[
\alpha = \prod_{i=1}^{n} q_i, \quad \text{and} \quad \sqrt{q_i} \neq \sqrt{q_j} \quad \text{if} \quad i \neq j.
\]

**Pf.** Let \( \alpha = \prod_{i=1}^{n} q_i \) be a reduced primary decomposition.
Consider \( \sqrt{q_i} \in \text{Max}(A) \) for any \( i \), and \( \sqrt{q_i} \neq \sqrt{q_j} \) if \( i \neq j \). Hence \( \sqrt{q_i}, \sqrt{q_j} \) are coprime. Therefore \( q_i \) and \( q_j \) are coprime. And so \( \bigcap_{i=1}^{n} q_i = \prod_{i=1}^{n} q_i \).

Suppose \( \mathfrak{A} = \prod_{i=1}^{m} q_i' \) such that \( q_i' \) is \( \mathfrak{p}_i' \)-primary and \( \mathfrak{p}_i' \neq \mathfrak{p}_j' \) if \( i \neq j \). Then again \( q_i' \) and \( q_j' \) are coprime, and so \( \mathfrak{A} = \bigcap_{i=1}^{m} q_i' \) is a primary decomposition. To get the uniqueness, it is enough to show this decomposition is reduced. If \( q_i' \supseteq \bigcap_{j=1}^{m} q_j' \), then \( \mathfrak{p}_i' \mid \prod_{j=1}^{m} q_j' \); and so \( \mathfrak{p}_i' \mid q_j' \) for some \( j \neq i \).

This implies \( \mathfrak{p}_j' \subseteq \mathfrak{p}_i' \) which contradicts \( \mathfrak{p}_j' \in \text{Max}(A) \).

Next, we will show that a reduced primary decomposition exists if \( A \) is Noetherian. Similar to the case of working with elements, we will define **irreducible ideals** and work with them.

**Def.** \( \mathfrak{a} \triangleleft A \) is called **irreducible** if \( \mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \), \( \mathfrak{b} \neq A, \mathfrak{c} \neq A \) imply either \( \mathfrak{a} = \mathfrak{b} \) or \( \mathfrak{a} = \mathfrak{c} \).
Next proposition gives us the connection between irreducible and primary ideals.

**Proposition.** An irreducible ideal is primary if \( A \) is Noetherian.

**Pf.** Suppose \( \mathfrak{a} \subseteq A \) is irreducible. Replacing \( A \) with \( A_{\mathfrak{a}} \), we can assume \( \mathfrak{a} \) is irreducible, and we have to show any zero-divisor is nilpotent. So suppose \( xy = 0 \) and \( y = 0 \). Now consider 
\[(0 : x) \subseteq (0 : x^2) \subseteq \cdots.\] Since \( A \) is Noetherian, \( \exists n \in \mathbb{Z}^+ \) s.t. 
\[(0 : x^n) = (0 : x^{n+1}).\]

**Claim.** \( \langle x^n \rangle \cap \langle y \rangle = 0 \)

**Pf.** Suppose \( z \in \langle x^n \rangle \cap \langle y \rangle \). Then \( z = x^n a = y a' \). Then 
\[xz = xya' = 0; \text{ and so } x^{n+1} a = xz = 0 \text{ which implies } a \in (0 : x^{n+1}) = (0 : x^n); \text{ therefore } z = x^n a = 0. \]

Since \( 0 \) is irreducible and \( y \neq 0 \), by the above claim \( x^n = 0 \) and so \( x \) is nilpotent and claim follows. \( \blacksquare \)
Theorem. In a Noetherian ring, any proper ideal has a primary decomposition.

Proof. By the previous proposition, it is enough to show any proper ideal can be written as an intersection of finitely many irreducible ideals. Let

\[ \Sigma := \{ \mathfrak{a} \neq 0 \mid \mathfrak{a} \text{ cannot be written as an intersection of finitely many irreducible ideals} \} \]

If \( \Sigma \neq \emptyset \), then it has a maximal element as \( A \) is Noetherian.

Say \( \mathfrak{a} \in \Sigma \) is a maximal element of \( \Sigma \). Then \( \mathfrak{a} \) is not irreducible.

And so \( \exists \mathfrak{b}, \mathfrak{c} \in A \) s.t. \( \mathfrak{b} \not\subset \mathfrak{a} \) and \( \mathfrak{c} \not\subset \mathfrak{a} \) and \( \mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \).

As \( \mathfrak{a} \) is maximal in \( \Sigma \), by (1) and (2) \( \exists \mathfrak{b}, \mathfrak{c} \notin \Sigma \). Therefore

\[ \exists \] irreducible ideals \( \mathfrak{q}_1, \ldots, \mathfrak{q}_m \) and \( \mathfrak{q}_{m+1}, \ldots, \mathfrak{q}_n \) s.t.

\[ \mathfrak{b} = \bigcap_{i=1}^{m} \mathfrak{q}_i \quad \text{and} \quad \mathfrak{c} = \bigcap_{i=m+1}^{n} \mathfrak{q}_i. \]

Therefore \( \mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} = \bigcap_{i=1}^{n} \mathfrak{q}_i \) can be written as an intersection of finitely many irreducible ideals, which is a contradiction. \( \square \)

Corollary. If \( A \) is Noetherian, then \( \text{Spec}(A) \) has only finitely many minimal elements \( \mathfrak{p}_1, \ldots, \mathfrak{p}_n \); in particular \( \text{Spec}(A) = \mathfrak{p}_1 \uplus \cdots \uplus \mathfrak{p}_n \).
Pf. Since $A$ is Noetherian, $0$ is decomposable. So

$$\{\text{minimal elements of } \text{Spec}(A)\} = \{\text{minimal elements of } \text{Ass}(0)\} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}.$$

Suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_m = V(\mathfrak{a})$ and $\mathfrak{p} \in \text{Spec}(A)$. Then

$$\exists i, \mathfrak{p}_i \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \mid \mathfrak{p}_i \mid \mathfrak{a} \Rightarrow \mathfrak{p} \in V(\mathfrak{a}); \text{ and so}$$

$$\mathfrak{p}_1, \ldots, \mathfrak{p}_m = \text{Spec}(A).$$