Lecture 10: Integral extension

Friday, April 20, 2018 10:41 PM

<u>Proposition</u> Suppose B/A is an integral extension.

(a) For $\mathfrak{b} \lhd B$, $A/\mathfrak{b}^c \hookrightarrow B/\mathfrak{b}$ is integral.

(b) $S^{-1}A \hookrightarrow S^{-1}B$ is integral, where $S \subseteq A$ is multip. closed.

<u>Pf.</u> (a) For $b \in B$, suppose $b^n + a_{n-1} b^{n-1} + \dots + a_0 = 0$ for $a_i \in A$. And let $\pi : B \to B/\mathfrak{b}$ be the natural quotient map. Then

$$\pi(b)^n + \pi(a_{n-1})\,\pi(b)^{n-1} + \dots + \pi(a_0) = 0;$$ and claim follows.

(b) For $\frac{b}{s} \in S^{-1}B$, suppose $b^n + a_{n-1} b^{n-1} + \dots + a_0 = 0$ for $a_i \in A$.
Then $\left(\frac{b}{s}\right)^n + \left(\frac{a_{n-1}}{s}\right)\left(\frac{b}{s}\right)^{n-1} + \dots + \left(\frac{a_0}{s^n}\right) = 0$ and $\frac{a_i}{s^{n-i}} \in S^{-1}A$. $\blacksquare$

<u>Proposition</u>. Suppose B/A is a ring extension, and C is the integral closure of A in B. For a multiplicative subset S of A, $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

<u>Pf.</u> We have already proved that $S^{-1}C/S^{-1}A$ is integral. Suppose b/s is integral over $S^{-1}A$. Then $\left(\frac{b}{s}\right)^n + \left(\frac{a_{n-1}}{s_{n-1}}\right)\left(\frac{b}{s}\right)^{n-1} + \dots + \left(\frac{a_0}{s_0}\right) = 0$

Let $s' := s_0 \cdot s_1 \cdot \dots \cdot s_{n-1}$. So

$$(s'b)^n + \frac{s'}{s_{n-1}} \cdot s \cdot a_{n-1}\,(s'b)^{n-1} + \dots + \frac{(s')^{n-i}}{s_i} \cdot s^{n-i} \cdot a_i\,(s'b)^i + \dots + \frac{(s')^n}{s_0} \cdot s^n a_0 = 0,$$

Lecture 10: Being integrally closed is a local property

Tuesday, April 24, 2018 9:21 AM

which implies $s'b$ is integral over A, and so $s'b = c \in C$.

Hence $\frac{b}{s} = \frac{s'b}{s's} = \frac{c}{s's} \in S^{-1}C$. ∎

<u>Corollary</u>. Suppose A is an integral domain. Then TFAE:

(a) A is integral closed.

(b) $\forall \mathfrak{p} \in \mathrm{Spec}(A)$, $A_{\mathfrak{p}}$ is integrally closed.

(c) $\forall \mathfrak{m} \in \mathrm{Max}(A)$, $A_{\mathfrak{m}}$ is integrally closed.

<u>Pf.</u> (a) $\Rightarrow$ (b) Let k be the field of fractions of A. Then by assumption the integral closure of A in k is A. And so the integral closure of $A_{\mathfrak{p}}$ in $S_{\mathfrak{p}}^{-1}k = k$ is $A_{\mathfrak{p}}$; claim follows.

(b) $\Rightarrow$ (c) is clear as $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A)$.

(c) $\Rightarrow$ (a) Let C be the integral closure of A in k. Then

$S_{\mathfrak{m}}^{-1} C = S_{\mathfrak{m}}^{-1} A$ for any $\mathfrak{m} \in \mathrm{Max}(A)$. And so

$S_{\mathfrak{m}}^{-1}(C/A) = 0$ for any $\mathfrak{m} \in \mathrm{Max}(A)$; this implies $C/A = 0$, and

$A = C$. ∎

Lecture 10: Integral extension and fields

Monday, April 23, 2018 12:31 AM

<u>Lemma</u>. Suppose B/A is an integral extension, and B is an integral domain. Then A is a field $\Leftrightarrow$ B is a field.

<u>Pf.</u> ($\Rightarrow$) $\forall\, b \in B$ ($b \neq 0$), $A[b]$ is a finite-dimensional A-algebra. Let $l_b : A[b] \to A[b]$, $l_b(v) = bv$. Then l_b is an inject. A-linear map; so, as $\dim_A A[b] < \infty$, l_b is surjective. And so $\exists\, b'$, $l_b(b') = 1 \Rightarrow b \in B^{\times} \Rightarrow B$ is a field.

($\Leftarrow$) $\forall\, a \in A$ ($a \neq 0$), $\exists\, a^{-1} \in B$; as B/A is integral, $\exists\, a_0, \ldots, a_{n-1} \in A$ s.t. $(a^{-n}) + a_{n-1}(a^{-n+1}) + \cdots + a_0 = 0$, which implies

$$a^{-1} = -(a_{n-1} + a_{n-2} \cdot a + \cdots + a_0 \cdot a^{n-1}) \in A. \quad \blacksquare$$

<u>Cor.</u> Suppose $f : A \hookrightarrow B$ is integral. Then

$$f^*(\mathfrak{q}) \in \operatorname{Max} A \iff \mathfrak{q} \in \operatorname{Max} B.$$

<u>Pf.</u> $A/f^*(\mathfrak{q}) \hookrightarrow B/\mathfrak{q}$ is integral and $B/\mathfrak{q}$ is an integral domain. So $A/f^*(\mathfrak{q})$ is a field $\Leftrightarrow$ $B/\mathfrak{q}$ is a field; and claim follows. $\blacksquare$

Lecture 10: Integral extension, maximal ideals, fibers

Thursday, April 19, 2018 10:55 PM

__Proposition__. $f: A \hookrightarrow B$ integral implies $f^*: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is onto.

__Pf.__ For $\mathfrak{p} \in \mathrm{Spec}(A)$, let $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$. Then $A_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{p}}^{-1} B$ is integral.

And so $f_{\mathfrak{p}}^*(\mathrm{Max}\, S_{\mathfrak{p}}^{-1} B) \subseteq \mathrm{Max}\, A_{\mathfrak{p}} = \{S_{\mathfrak{p}}^{-1}\mathfrak{p}\}$. So

$\exists\, \mathfrak{q} \in \mathrm{Spec}\, B$ s.t. $\mathfrak{q} \cap S_{\mathfrak{p}} = \emptyset$ and $S_{\mathfrak{p}}^{-1}\mathfrak{q} \cap A_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}\mathfrak{p}$

And so $\underbrace{\mathfrak{q} \cap A}_{\mathfrak{q}^c} \subseteq \mathfrak{p}$ and $\mathfrak{q} \supseteq \mathfrak{p}$; which implies $f^*(\mathfrak{q}) = \mathfrak{q}^c = \mathfrak{p}$.

$\blacksquare$