Lecture 12: Minimal poly and integrally closed rings

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<u>Lemma</u>. A: integral domain, integrally closed, and F is its field of fract.

B: integral domain, $B \supseteq A$, and $b \in B$ is integral over A. Suppose

$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in F[x]$ is the minimal polynomial of b over F.

Then $a_0, \ldots, a_{n-1} \in \sqrt{\mathfrak{a}}$.

<u>Pf</u>. Let E be the normal closure of $F[b]$ over F, and C be the integral closure of A in E. Since b is integral over $\mathfrak{a}$, $\exists\, g(x) = x^m + a'_{m-1}x^{m-1} + \cdots + a'_0$ s.t. $g(b) = 0$ and $a'_i \in \mathfrak{a}$. And so $f(x) \mid g(x)$. $(*)$ As E/F is a normal closure and $f(b) = 0$, $\exists\, \beta_i \in E$ s.t. $f(x) = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_n)$. Therefore by $(*)$, $g(\beta_i) = 0$, which implies β_i's are integral over $\mathfrak{a}$. Hence coefficients of f (except the leading coeff.) are integral over $\mathfrak{a}$. As A is integrally closed, we deduce $a_i \in A$ and a_i is integral over $\mathfrak{a}$.

Therefore by a result that we proved in the previous lecture $a_i \in \sqrt{\mathfrak{a}}$. ■

<u>Proposition</u>. A: integral domain, integrally closed; $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \in \operatorname{Spec} A$

$$\begin{array}{c} \mathfrak{p}_1 \\ \uparrow \\ \mathfrak{q}_1 \in \operatorname{Spec} B \end{array}$$

B: integral domain; B/A: integral.

$\mathfrak{p}_1 = \mathfrak{q}_1 \cap A$. Then $\exists\, \mathfrak{q}_0 \subseteq \mathfrak{q}_1$ st. $\mathfrak{q}_0 \cap A = \mathfrak{p}_0$.

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Pf. Recall that $\{\mathfrak{q} \in \operatorname{Spec}(B) \mid \mathfrak{q} \subseteq \mathfrak{q}_1\} \underset{e}{\overset{c}{\rightleftarrows}} \operatorname{Spec} B_{\mathfrak{q}_1}$.

and $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \subseteq \mathfrak{p}_1\} \underset{e}{\overset{c}{\rightleftarrows}} \operatorname{Spec} A_{\mathfrak{p}_1}$. So we have to show

$\operatorname{Spec} B_{\mathfrak{q}_1} \xrightarrow{g^*} \operatorname{Spec} A_{\mathfrak{p}_1}$ is onto. To show $S_{\mathfrak{p}_1}^{-1} \mathfrak{p}_0$ is in the

image of g^*, we have to show $(S_{\mathfrak{p}_1}^{-1} \mathfrak{p}_0)^{ec} = S_{\mathfrak{p}_1}^{-1} \mathfrak{p}_0$.

Clearly $S_{\mathfrak{p}_1}^{-1} \mathfrak{p}_0 \subseteq (S_{\mathfrak{p}_1}^{-1} \mathfrak{p}_0)^{ec}$. So we have to show

$$\mathfrak{p}_0 B_{\mathfrak{q}_1} \cap A_{\mathfrak{p}_1} \subseteq \mathfrak{p}_0 A_{\mathfrak{p}_1}.$$

(Contrary assumption)

Suppose for $a \in A$, $s \in S_{\mathfrak{p}_1}$, $\frac{a}{s} \in \mathfrak{p}_0 B_{\mathfrak{q}_1} \setminus \mathfrak{p}_0 A_{\mathfrak{p}_1}$. Hence

$$\frac{a}{s} = \sum_{i=1}^{\ell} p_i \cdot \frac{b_i}{s_i} \quad \text{for } p_i \in \mathfrak{p}_0,\ b_i \in B,\ s_i \in S_{\mathfrak{q}_1}.$$

$$\Rightarrow \underbrace{(\textstyle\prod s_i)}_{s'} a = s \sum_{i=1}^{\ell} p_i \cdot \left(\frac{\prod s_i}{s_i}\right) b_i \in \mathfrak{p}_0^e$$

$\Rightarrow$ $s'a$ is integral over $\mathfrak{p}_0$, and $s' \in S_{\mathfrak{q}_1}$.

$\Rightarrow$ the minimal poly. of $s'a$ over the field of fractions F of A

is of the form $f(x) := x^n + p'_{n-1} x^{n-1} + \cdots + p'_0$ where $p'_i \in \mathfrak{p}_0$.

So the minimal polynomial of s' is $g(x) = \frac{1}{a^n} f(ax)$. Since

$s' \in B$ is integral over A, $g(x) \in A[x]$. Hence $\frac{1}{a^n} f(ax) \in A[x]$,

which implies $p'_{n-i} = a^i \cdot p''_{n-i}$ for some $p''_{n-i} \in A$. By the contrary

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assumption and $p_i' \in \mathfrak{p}_0$, we deduce $p_i'' \in \mathfrak{p}_0$; and so

$$g(x) = \frac{1}{a^n} f(ax) = x^n + \sum_{i=0}^{n-1} \frac{p_i'}{a^n} (ax)^i = x^n + p_{n-1}'' x^{n-1} + \cdots + p_0'' .$$

Hence s' is integral over $\mathfrak{p}_0$; this implies $s' \in \sqrt{\mathfrak{p}_0^e} \subseteq \mathfrak{q}_1$

which is a contradiction. ∎

Going-Down Theorem. A, B : integral domain ; A : integrally closed ;

B/A : integral. $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \in \operatorname{Spec} A$;

and $\mathfrak{q}_m \subsetneq \cdots \subsetneq \mathfrak{q}_n \in \operatorname{Spec} B$ s.t. $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.

Then $\exists\ \mathfrak{q}_0, \ldots, \mathfrak{q}_{m-1}$ s.t. $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m$ and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.

Pf. It is an immediate corollary of the previous proposition. ∎

An alternative approach.

Theorem. A : integral domain, integrally closed, with field of fract. F.

E/F : normal field extension.

B : integral closure of A in E. $(f : A \hookrightarrow B)$

$\Rightarrow \forall \mathfrak{p} \in \operatorname{Spec}(A)$, $\operatorname{Aut}(E/F) \curvearrowright (f^*)^{-1}(\mathfrak{p})$ transitively.

Pf.

Initial Observ. $\forall b \in B$, $\min(b; F) \in A[x]$; and $\forall \sigma \in \operatorname{Aut}(E/F)$, $\min(b; F)$ $= \min(\sigma(b); F)$. Hence $\sigma(B) = B$. If $\mathfrak{q} \cap A = \mathfrak{p}$, then $\sigma(\mathfrak{q}) \cap A = \mathfrak{p}$.

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Hence $\mathrm{Aut}(E/F) \curvearrowright (f^*)^{-1}(\mathfrak{p})$.

Let $G := \mathrm{Aut}(E/F)$; then $F \subseteq \mathrm{Fix}(G) \subseteq E$; $E/\mathrm{Fix}(G)$ is a Galois extension, and $\mathrm{Fix}(G)/F$ is purely inseparable.

(We will continue next time.)