In the previous lecture we proved the following technical theorem:

\[ \Omega: \text{algebraically closed field}; \]
\[ A_0: \text{subring of a field } F; \quad \phi: A_0 \to \Omega \quad \text{ring hom.}; \]
\[ \Sigma := \{ (A, \phi) \mid A_0 \subseteq A \subseteq F; \quad A \xrightarrow{\phi} \Omega \} \quad \text{subring} \]
\[ (A_1, \phi_1) \preceq (A_2, \phi_2) \quad \text{if} \quad A_1 \subseteq A_2 \quad \text{and} \quad \phi_2 \mid_{A_1} = \phi_1. \]

Then \( \Sigma \) has a maximal element; if \( (B, \theta) \) is maximal in \( \Sigma \), then \( B \) is a valuation ring, its field of fractions is \( F \), and \( \ker \theta \) is the maximal ideal of \( B \).

**Proposition.** Let \( A \) be an integral domain with field of fractions \( F \).

The integral closure of \( A \) in \( F \) is:

\[ \text{integral closure of } A \text{ in } F = \bigcap_{A \subseteq B \subseteq F \atop B \text{ valuation ring}} B \]

**Proof.** Since any valuation ring is integrally closed and by the tech. theorem \( \exists A \subseteq B \subseteq F \), \( B \) is a valuation ring, \( \text{RHS} \subseteq \text{LHS} \).

- Suppose \( f \in F \) is not integral over \( A \). So \( f \notin A[f^{-1}] \); this implies \( \exists \delta \in \text{Max} (A[f^{-1}]) \text{ s.t. } f^{-1} \notin \delta \). Let \( \Omega \) be an alg.
Lecture 17: Lifting ring homomorphisms

Monday, May 7, 2018 8:35 AM

closure of $A\ell_f^{\downarrow}/_{\text{fin}}$, and $A\ell_f^{\downarrow}/_{\text{fin}} \twoheadrightarrow A\ell_f^{\downarrow}/_{\text{fin}} \twoheadrightarrow \Omega$. \\

Then by the technical theorem $\exists$ a valuation ring $B$ and $\theta : B \rightarrow \Omega$ st. $A\ell_f^{\downarrow} \subseteq B$ and $\ker \theta \supseteq \text{fin} \cap f$. \\

Hence $f \notin B$. $\blacksquare$

The 2$^{\text{nd}}$ important consequence of the technical theorem is:

**Theorem.** $A$ : integral domain \\
$B$ : integral domain, i.e. $A$ -alg. ; $A \subseteq B$. \\
$b_0 \in B \setminus \Omega$. \\

$\Rightarrow \exists a_o := a_0 (b_0) \in A$ st. $\forall \phi \in \text{Hom}(A, \Omega), \phi (a_o) \neq 0$ \\

$\exists \theta \in \text{Hom}(B, \Omega)$ st. $\theta|_A = \phi$ and $\theta(b_0) \neq 0$.

**Proof.** We proceed by induction on the number of generators of $B$ as an $A$-algebra. So it is enough to prove the case $B = A\ell_f^{\downarrow}$.

**Case 1.** $\beta$ is transcendental over $A$.

Then $b_0 = c_n \beta^n + c_{n-1} \beta^{n-1} + \cdots + c_0$ where $c_i \in A$. Let $a_o := c_n$. \\

\[ \text{proof continued...} \]
Then \( \phi(c_n) t^n + \phi(c_{n-1}) t^{n-1} + \cdots + \phi(c_0) \) is a non-zero poly. in \( \Omega[H] \) if \( \phi(c_0) \neq 0 \). As \( |\Omega| = \omega \), \( \exists \alpha_0 \in \Omega \) which is not a zero of this poly. Then

\[
\theta \left( \sum d_i \beta^i \right) := \sum \phi(d_i) \alpha_i^j \text{ satisfies the needed conditions.}
\]

**Case 2.** \( \beta \) is algebraic over \( A \).

Then \( B = A[\beta] / A \) is algebraic. So \( \exists a_i' \) and \( a_i'' \in A \) s.t.

\[
(\text{I}) \quad a_n' \beta^n + a_{n-1}' \beta^{n-1} + \cdots + a_0' = 0 \quad \text{and}
\]

\[
(\text{II}) \quad a_m'' b_m^{-m} + a_{m-1}' b_m^{-(m-1)} + \cdots + a_0'' = 0.
\]

Let \( a_0 := a_n' a_m'' \). If \( \phi(a_n' a_m'') \neq 0 \) for some \( \phi : A \to \Omega \), then \( \phi \) has a lift \( \hat{\phi} : A[\frac{1}{a_n' a_m''}] \to \Omega \).

Consider \( A[\frac{1}{a_n' a_m''}] \) as a subring of the field \( E \) of fractions of \( B \), and use the technical theorem to deduce:

\( \exists \) a valuation ring \( C \) with field of fractions \( E \), and a lift

\[
\hat{\phi} : C \to \Omega \text{ of } \hat{\phi}.
\]

By (I) and (II), \( \beta \) and \( b_m^{-1} \) are integral over \( A[\frac{1}{a_n' a_m''}] \). And so
$p$ and $b_0$ are integral over $C$. Since $C$ is integrally closed, $p, b_0 \in C$.

$\Rightarrow$ (1) $B \subseteq C$ (2) $b_0 \in C^x \Rightarrow \phi|_B : B \to \Omega$ is a lift of $\phi$ and $\phi(b_0) \neq 0$.

**Theorem (1st version of Hilbert’s Nullstellensatz)**

$k$: field. $B$: f.g. $k$-algebra.

If $B$ is a field, then $B/k$ is a finite extension.

**Pf.** Let $\Omega$ be an algebraic closure of $k$, and $\phi : k[C] \to \Omega$ be an embedding. Let $b_0 = 1$. Then $\exists a_0 \in k$ s.t. if $\phi(a_0) \neq 0$,

then $\phi$ has a lift $\hat{\phi} : B \to \Omega$. But, since $\phi$ is an embedding, $\phi(a_0) \neq 0$. So $\exists$ a lift $\hat{\phi} : B \to \Omega$ of $\phi : k[C] \to \Omega$.

Since $B$ is a field, $\hat{\phi}$ is an embedding. And so $B/k$ is an algebraic extension. Since $B$ is a f.g. $k$-algebra,

$B/k$ is a finite extension.