Lecture 18: The 2nd version of Hilbert's Nullstellensatz

Wednesday, May 9, 2018 11:15 AM

In the previous lecture we proved

Theorem (1st version of Hilbert's Nullstellensatz)

k: field; B: f.g. k-alg and field $\Rightarrow B/k$ is a finite extension.

Here is our 2nd version:

Theorem (2nd version of Hilbert's Nullstellensatz)

Suppose k is an algebraically closed field. Let $A := k[x_1, \ldots, x_n]/\mathfrak{a}$.

where $\mathfrak{a} \trianglelefteq k[x_1, \ldots, x_n]$. Then $\operatorname{Max} A = \{\mathfrak{m}_p/\mathfrak{a} \mid p \in X(\mathfrak{a})\}$

where $X(\mathfrak{a}) := \{p \in k^n \mid \forall f(x) \in \mathfrak{a}, f(p) = 0\}$,

(Common zeros of elements of $\mathfrak{a}$.)

and $\mathfrak{m}_p := \langle x_1 - p_1, \ldots, x_n - p_n \rangle \trianglelefteq k[x_1, \ldots, x_n]$, (here

$p = (p_1, \ldots, p_n)$.)

(So there is a bijection between

$$\operatorname{Max}(k[x_1, \ldots, x_n]/\mathfrak{a}) \quad \text{and} \quad X(\mathfrak{a}).)$$

Before we get to the proof of the 2nd version of Hilbert's Nullstellensatz let's discuss basic properties of $\mathfrak{a} \longmapsto X(\mathfrak{a})$ and its parallel

Lecture 18: 2nd version of Hilbert's Nullstellensatz

Wednesday, May 9, 2018 8:12 AM

with $\mathfrak{a} \longmapsto V(\mathfrak{a}) \subseteq \operatorname{Spec}(k[x_1,\dots,x_n])$.

- $\mathfrak{a} = \langle f_1,\dots,f_m\rangle \Rightarrow X(\mathfrak{a}) = X(f_1,\dots,f_m)$.
- $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow X(\mathfrak{b}) \subseteq X(\mathfrak{a})$ $\quad V(\mathfrak{b}) \subseteq V(\mathfrak{a})$
- $X(\mathfrak{a}) = X(\sqrt{\mathfrak{a}})$ $\quad$ (notice that $f(p)^n = 0$ implies $f(p) = 0$
  $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$ $\quad$ as k has no nilpotent element.)
- $X(\mathfrak{m}_p) = \{p\}$ $\quad V(\mathfrak{m}_p) \underset{?}{=} \{\mathfrak{m}_p\}$
- $X(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} X(\mathfrak{a}_i)$. $\quad V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$
- $X(\mathfrak{a}_1\cap\dots\cap\mathfrak{a}_m) \supseteq \bigcup_{i=1}^m X(\mathfrak{a}_i)$ $\quad V(\mathfrak{a}_1\cap\dots\cap\mathfrak{a}_m) = \bigcup_{i=1}^m V(\mathfrak{a}_i)$.
- A subset of k^n is called <u>k-closed</u> if it is $X(\mathfrak{a})$ for some $\mathfrak{a}$.

<u>Pf of 2^{nd} version of Hilbert's Nullstellensatz.</u>

- $\forall p \in X(\mathfrak{a})$, let $\tilde{e}_p : k[x_1,\dots,x_n] \longrightarrow k$ be the evaluation map

at p. Hence $\ker \tilde{e}_p \supseteq \mathfrak{m}_p$ and $\mathfrak{a} \subseteq \ker \tilde{e}_p$. If $f \in \ker \tilde{e}_p$,

writing the Taylor expansion of f at p, we deduce

$$f(x) = \underbrace{f(p)}_{0} + \sum \partial_i f(p)\,(x_i - a_i) + \text{higher order terms} \in \mathfrak{m}_p .$$

$\Rightarrow \ker \tilde{e}_p = \mathfrak{m}_p$. Hence ① $\mathfrak{a} \subseteq \mathfrak{m}_p$ ② $k[x_1,\dots,x_n]/\mathfrak{m}_p \simeq k$.

Lecture 18: 3rd version of Hilbert's Nullstellensatz

Wednesday, May 9, 2018 8:28 AM

$\Rightarrow \mathfrak{m}_p/\mathfrak{a} \in \operatorname{Max} A$. (② show also $V(\mathfrak{m}_p) = \{\mathfrak{m}_p\}$.)

- Suppose $\mathfrak{m} \in \operatorname{Max} A$. Let $B := A/\mathfrak{m}$.

 Notice that $\exists\, \tilde{\mathfrak{m}} \in \operatorname{Max} k[x_1, \dots, x_n]$ s.t. $\mathfrak{m} = \tilde{\mathfrak{m}}/\mathfrak{a}$; and so

 $B \simeq k[x_1, \dots, x_n]/\tilde{\mathfrak{m}}$ is a f.g. k-algebra and a field.

 Hence by the 1^{st} version of Hilbert's Nullstellensatz theorem,

 B/k is a finite extension. Since k is algebraically closed,

 we deduce that $B = k$. Therefore $\exists\, p_1, \dots, p_n \in k$ s.t.

 $x_i + \tilde{\mathfrak{m}} = p_i + \tilde{\mathfrak{m}}$, which implies $\mathfrak{m}_p \subseteq \tilde{\mathfrak{m}}$. And so $\tilde{\mathfrak{m}} = \mathfrak{m}_p$ as

 $\mathfrak{m}_p$ is maximal. ∎

<u>Thm</u> (3^{rd} version of Hilbert's Nullstellensatz)

Suppose k is algebraically closed field and $\mathfrak{a} \underset{\neq}{\trianglelefteq} k[x_1, \dots, x_n]$. Then

$X(\mathfrak{a}) \neq \emptyset$.

<u>Pf</u>. $\exists\, \mathfrak{m} \in \operatorname{Max} k[x_1, \dots, x_n]$ s.t. $\mathfrak{a} \subseteq \mathfrak{m}$. By the 2^{nd} version

of Hilbert's Nullstellensatz $\exists\, p$ s.t. $\mathfrak{m} = \mathfrak{m}_p$. $\Rightarrow \mathfrak{a} \subseteq \mathfrak{m}_p$

$\Rightarrow p \in X(\mathfrak{a})$. ∎

Lecture 18: 4th version of Hilbert's Nullstellensatz

Wednesday, May 9, 2018 8:45 AM

<u>Theorem</u> (4^{th} version of Hilbert's Nullstellensatz)

Suppose k is algebraically closed, $\mathfrak{a} \underset{\neq}{\trianglelefteq} k[x_1, \ldots, x_n]$. Then

$$\sqrt{\mathfrak{a}} = I(X(\mathfrak{a})),$$

where $I(Y) = \{ f(x) \in k[x_1, \cdots, x_n] \mid f|_Y = 0 \}$ for some $Y \subseteq k^n$.

<u>Pf.</u> • $f \in \sqrt{\mathfrak{a}} \Rightarrow f^m \in \mathfrak{a}$ for some $m \in \mathbb{Z}^+$

$\Rightarrow f^m|_{X(\mathfrak{a})} = 0 \Rightarrow f|_{X(\mathfrak{a})} = 0 \Rightarrow f \in I(X(\mathfrak{a}))$.

• Suppose $f \in I(X(\mathfrak{a})) \setminus \sqrt{\mathfrak{a}} \Rightarrow S_f \cap \mathfrak{a} = \emptyset$ where $S_f = \{1, f, f^2, \cdots\}$.

• Notice that $S_f^{-1} k[x_1, \cdots, x_n] \simeq k[x_1, \ldots, x_n, x_{n+1}] / \langle 1 - x_{n+1} f \rangle$, $\circledast$

and $S_f^{-1} k[x_1, \cdots, x_n] / S_f^{-1}\mathfrak{a} \simeq k[x_1, \ldots, x_n, x_{n+1}] / \langle 1 - x_{n+1} f \rangle + \mathfrak{a}[x_{n+1}]$

(here $\mathfrak{a}[x_{n+1}]$ is the extension of $\mathfrak{a}$ to an ideal of $k[x_1, \cdots, x_{n+1}]$.)

And so $\langle 1 - x_{n+1} f \rangle + \mathfrak{a}[x_{n+1}]$ is a <u>proper</u> ideal of $k[x_1, \cdots, x_{n+1}]$. Therefore by the 3^{rd} version of Hillbert's Nullstellensatz, $X(\langle 1 - x_{n+1} f \rangle + \mathfrak{a}[x_{n+1}]) \neq \emptyset$. Say (p, p_{n+1}) is in

Lecture 18: 4th version of Hilbert's Nullstellensatz

Friday, May 11, 2018 8:48 AM

$X(\langle 1-x_{n+1}f\rangle + \mathfrak{a}[x_{n+1}])$. Then

① $p \in X(\mathfrak{a})$ ② $1 = p_{n+1} f(p)$.

Since $p \in X(\mathfrak{a})$ and $f \in I(X(\mathfrak{a}))$, $f(p) = 0$ which contradicts ②

$\circledast$. To show this we prove the following lemma:

<u>Lemma</u>. Suppose D is an integral domain and $d_0 \in D \setminus \{0\}$. Then

$$D[1/d_0] \simeq D[x]/\langle d_0 x - 1\rangle.$$

<u>Pf</u>. Let $\tilde{e}_{1/d_0} : D[x] \longrightarrow D[1/d_0]$, $\tilde{e}_{1/d_0}(f(x)) := f(1/d_0)$.

Then $\tilde{e}_{1/d_0}$ is a ring hom. and $d_0 x - 1 \in \ker \tilde{e}_{1/d_0}$, and $\tilde{e}_{1/d_0}$ is onto. So $\exists$ an onto ring hom.

$$e_{1/d_0} : D[x]/\langle d_0 x - 1\rangle \longrightarrow D[1/d_0].$$

On the other hand, let $\bar{x} := x + \langle d_0 x - 1\rangle \in D[x]/\langle d_0 x - 1\rangle$.

And so $d_0 \cdot \bar{x} = 1$. Hence by the universal property of localization

$\exists\ \phi : D[1/d_0] \longrightarrow D[x]/\langle d_0 x - 1\rangle$, $\phi\left(\frac{d}{d_0^m}\right) = dx^m + \langle d_0 x - 1\rangle$.

And clearly ϕ and e_{1/d_0} are inverse of each other. ∎