In the previous lecture we proved

**Theorem (1st version of Hilbert's Nullstellensatz)**

\[ k : \text{field} ; \ B : \text{f.g.} \ k-\text{alg} \text{ and field} \Rightarrow B/\k \text{ is a finite extension} . \]

Here is our 2nd version:

**Theorem (2nd version of Hilbert's Nullstellensatz)**

Suppose \( k \) is an algebraically closed field. Let \( A := \frac{k[x_1, \ldots, x_n]}{I} \).

where \( I \) is an ideal of \( k[x_1, \ldots, x_n] \). Then \( \text{Max} A = \{ \mathfrak{m}_p / I \mid p \in X(I) \} \).

where \( X(I) := \{ x \in k^n \mid \forall f(x) \in I, f(p) = 0 \} \), (Common zeros of elements of \( I \)).

and \( \mathfrak{m}_p := \langle x_1 - p_1, \ldots, x_n - p_n \rangle \triangleleft k[x_1, \ldots, x_n] \), (here \( p = (p_1, \ldots, p_n) \)).

(So there is a bijection between \( \text{Max} k[x_1, \ldots, x_n]/I \) and \( X(I) \).)

Before we get to the proof of the 2nd version of Hilbert's Nullstellensatz, let's discuss basic properties of \( I \mapsto X(I) \) and its parallel
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with \( \mathfrak{a} \rightarrow V(\mathfrak{a}) \subseteq \text{Spec } (k[x_1, \ldots, x_n]) \).

1. \( \mathfrak{a} = \langle f_1, \ldots, f_m \rangle \Rightarrow X(\mathfrak{a}) = X(\langle f_1, \ldots, f_m \rangle) \).

2. \( \mathfrak{a} \subseteq \mathfrak{b} \Rightarrow X(\mathfrak{a}) \subseteq X(\mathfrak{b}) \)

\( V(\mathfrak{a}) \subseteq V(\mathfrak{b}) \)

3. \( X(\mathfrak{a}) = X(\sqrt{\mathfrak{a}}) \) (notice that \( \mathfrak{a} \cap \mathfrak{b} = 0 \) implies \( \mathfrak{a} \cap \mathfrak{b} = 0 \) as \( k \) has no nilpotent element.)

\( V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}}) \)

4. \( X(\sum_{i=1}^{m} \mathfrak{a}_i) = \bigcap_{i=1}^{m} X(\mathfrak{a}_i) \)

\( V(\sum_{i=1}^{m} \mathfrak{a}_i) = \bigcap_{i=1}^{m} V(\mathfrak{a}_i) \)

5. \( X(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_m) \supseteq \bigcup_{i=1}^{m} X(\mathfrak{a}_i) \)

\( V(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_m) = \bigcup_{i=1}^{m} V(\mathfrak{a}_i) \)

A subset of \( k^n \) is called \( k \)-closed if it is \( X(\mathfrak{a}) \) for some \( \mathfrak{a} \).

Proof of 2nd version of Hilbert's Nullstellensatz.

Theorem: For every \( \mathfrak{a} \subseteq X(\mathfrak{a}) \), let \( \mathfrak{e}_p : k[x_1, \ldots, x_n] \rightarrow k \) be the evaluation map at \( p \). Hence \( \ker \mathfrak{e}_p \supseteq \mathfrak{a} \), and \( \mathfrak{a} \subseteq \ker \mathfrak{e}_p \). If \( f \in \ker \mathfrak{e}_p \), writing the Taylor expansion of \( f \) at \( p \), we deduce

\( f(x) = f(p) + \sum \partial_i f(p) (x_i - a_i) + \text{ higher order terms } \in \mathfrak{a} \).

\( \Rightarrow \ker \mathfrak{e}_p = \mathfrak{a} \). Hence:

1. \( \mathfrak{a} \subseteq \mathfrak{a} \)
2. \( k[x_1, \ldots, x_n]/\mathfrak{a} \approx k \).
\[ \mathfrak{m}_T / \mathfrak{m} \in \text{Max } A. \quad (\emptyset \text{ show also } V(\mathfrak{m}_T) = \emptyset \text{ if } \mathfrak{m} \neq 0.) \]

Suppose \( \mathfrak{m} \in \text{Max } A \). Let \( B := A / \mathfrak{m} \).

Notice that \( \exists \mathfrak{m} \in \text{Max } k[x_1, \ldots, x_n] \text{ s.t. } \mathfrak{m} = \mathfrak{m} / \mathfrak{m}_T \); and so \( B = k[x_1, \ldots, x_n] / \mathfrak{m}_T \) is a f.g. \( k \)-algebra and a field.

Hence by the 1st version of Hilbert's Nullstellensatz theorem, \( B / k \) is a finite extension. Since \( k \) is algebraically closed, we deduce that \( B = k \). Therefore \( \exists p_1, \ldots, p_n \in k \) s.t.

\[ x_i + \mathfrak{m}_T = p_i + \mathfrak{m}_T, \text{ which implies } \mathfrak{m}_T \subseteq \mathfrak{m}_p. \text{ And so } \mathfrak{m}_T = \mathfrak{m}_p \text{ as } \mathfrak{m}_p \text{ is maximal}. \]

\[ \text{Thm (3rd version of Hilbert's Nullstellensatz)} \]

Suppose \( k \) is algebraically closed field and \( \mathfrak{a} \subseteq k[x_1, \ldots, x_n] \). Then

\[ X(\mathfrak{a}) \neq \emptyset. \]

\[ \text{Pf. } \exists \mathfrak{m} \in \text{Max } k[x_1, \ldots, x_n] \text{ s.t. } \mathfrak{a} \subseteq \mathfrak{m}. \text{ By the 2nd version of Hilbert's Nullstellensatz } \exists p \text{ s.t. } \mathfrak{m} = \mathfrak{m}_p \text{, } \Rightarrow \mathfrak{a} \subseteq \mathfrak{m}_p \]

\[ \Rightarrow p \in X(\mathfrak{a}). \]

\[ \blacksquare \]
Theorem (4th version of Hilbert’s Nullstellensatz)

Suppose $k$ is algebraically closed, $\mathcal{A} \subseteq k[\mathbf{x}_1, \ldots, \mathbf{x}_n]$. Then

$$\sqrt{\mathcal{A}} = I(\mathcal{X}(\mathcal{A})),$$

where $I(Y) = \{ f \in k[\mathbf{x}_1, \ldots, \mathbf{x}_n] \mid f|_Y = 0 \}$ for some $Y \subseteq k^n$.

Proof: $f \in \sqrt{\mathcal{A}} \Rightarrow f^m \in \mathcal{A}$ for some $m \in \mathbb{Z}^+$

$\Rightarrow \frac{f^m}{\mathcal{X}(\mathcal{A})} = 0 \Rightarrow \frac{f}{\mathcal{X}(\mathcal{A})} = 0 \Rightarrow f \in I(\mathcal{X}(\mathcal{A}))$.

Suppose $f \in I(\mathcal{X}(\mathcal{A})) \backslash \sqrt{\mathcal{A}} \Rightarrow S_f \cap \mathcal{A} = \emptyset$ where

$$S_f = \{ f^1, f^2, \ldots \}.$$

Notice that $S_f^{-1} k[\mathbf{x}_1, \ldots, \mathbf{x}_n] \cong k[\mathbf{x}_1, \ldots, \mathbf{x}_n, x_{n+1}] / \langle 1 - x_{n+1} f \rangle$, and $S_f^{-1} k[\mathbf{x}_1, \ldots, \mathbf{x}_n] / S_f^{-1} \mathcal{A} \cong k[\mathbf{x}_1, \ldots, \mathbf{x}_n, x_{n+1}] / \langle 1 - x_{n+1} f \rangle + \mathcal{A}[x_{n+1}]$.

(here $\mathcal{A}[x_{n+1}]$ is the extension of $\mathcal{A}$ to an ideal of $k[\mathbf{x}_1, \ldots, x_{n+1}]$)

And so $\langle 1 - x_{n+1} f \rangle + \mathcal{A}[x_{n+1}]$ is a proper ideal of $k[\mathbf{x}_1, \ldots, x_{n+1}]$. Therefore by the 3rd version of Hilbert’s Nullstellensatz, $\mathcal{X}(\langle 1 - x_{n+1} f \rangle + \mathcal{A}[x_{n+1}]) \neq \emptyset$. Say $(p, p_{n+1})$ is in
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$$X \langle 1 - x_{n+1} \rangle + \mathfrak{m} I(x_{n+1})$$. Then

1. $$\mathfrak{p} \in X(\mathbb{C})$$
2. $$1 = \frac{1}{\mathfrak{p}+1} f(\mathfrak{p})$$.

Since $$\mathfrak{p} \in X(\mathbb{C})$$ and $$f \in I(X(\mathbb{C}))$$, $$f(\mathfrak{p}) = 0$$, which contradicts 2.

To show this we prove the following lemma:

**Lemma.** Suppose $$D$$ is an integral domain and $$d_0 \in D \setminus \{0\}$$. Then

$$D[{1/d_0}] \cong D[x] / \langle d_0 x - 1 \rangle$$.

**Proof.** Let $$\tilde{\varepsilon}_{1/d_0} : D[x] \to D[{1/d_0}]$$, $$\tilde{\varepsilon}_{1/d_0}(f(x)) := f(1/d_0)$$. Then $$\tilde{\varepsilon}_{1/d_0}$$ is a ring hom. and $$d_0 x - 1 \in \ker \varepsilon_{1/d_0}$$, and $$\varepsilon_{1/d_0}$$ is onto. So $$\exists$$ an onto ring hom.

$$\varepsilon_{1/d_0} : D[x] / \langle d_0 x - 1 \rangle \to D[{1/d_0}]$$.

On the other hand, let $$\overline{x} = x + \langle d_0 x - 1 \rangle \in D[x] / \langle d_0 x - 1 \rangle$$. And so $$d_0 \cdot \overline{x} = 1$$. Hence by the universal property of localization

$$\exists \phi : D[{1/d_0}] \to D[x] / \langle d_0 x - 1 \rangle$$, $$\phi \left( \frac{d}{d_0} \right) = d \overline{x} + \langle d_0 x - 1 \rangle$$. And clearly $$\phi$$ and $$\varepsilon_{1/d_0}$$ are inverse of each other.

$$\square$$