Lecture 20: Noether normalization

Monday, May 14, 2018 1:12 AM

We were proving Noether's normalization lemma:

<u>Theorem</u>. k: field; A: f.g. k-algebra. Then $\exists\, \xi_1, \ldots, \xi_n \in A$ s.t.

① $\xi_1, \ldots, \xi_n$ are algebraically independent over k.

② $A / k[\xi_1, \ldots, \xi_n]$ is an integral extension.

We started its proof by induction on the number of generators of A. And we have addressed the base of induction. We also proved a technical lemma: $\forall f \in k[x_1, \ldots, x_n] \setminus \{0\}$, $\exists\, \phi \in \mathrm{Aut}_k(k[x_1, \ldots, x_n])$ s.t. the leading coeff. of $\phi(f)$ viewed as an element of $(k[x_1, \ldots, x_{n-1}])[x_n]$ is in $k^\times$.

<u>Induction step</u>. Suppose $A = k[\alpha_1, \ldots, \alpha_n]$. If α_i's are not algebraic over k, we are done. So suppose $f(\alpha_1, \ldots, \alpha_n) = 0$ for some $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n] \setminus \{0\}$. By the mentioned result $\exists\, \phi$ in $\mathrm{Aut}_k(k[x_1, \ldots, x_n])$ st. the leading coeff. of $\phi(f)$ viewed as an element of $(k[x_1, \ldots, x_{n-1}])[x_n]$ is in $k^\times$. So ϕ induces a k-isomorphism $\Phi : k[\alpha_1, \ldots, \alpha_n] \xrightarrow{\sim} k[\beta_1, \ldots, \beta_n]$ and $\phi(f)(\beta_1, \ldots, \beta_n) = 0$. Hence β_n is integral over $k[\beta_1, \ldots, \beta_{n-1}]$. By the induction hypothesis

Lecture 20: Noether normalization

Monday, May 14, 2018 1:23 AM

$\exists\, t_1, \dots, t_m \in k[\beta_1, \dots, \beta_{n-1}]$ s.t. (1) t_i's are algebraically independent

(2) $k[\beta_1, \dots, \beta_{n-1}]$ is integral over $k[t_1, \dots, t_m]$.

Since β_n is integral over $k[\beta_1, \dots, \beta_{n-1}]$, we get that

$k[\beta_1, \dots, \beta_n]$ is integral over $k[t_1, \dots, t_m]$; and claim follows. ■

<u>Corollary</u>. A: f.g. k-algebra

$\Rightarrow \exists$ an integral embedding $k[x_1, \dots, x_n] \hookrightarrow A$ of the ring of polynomials.

In this case, $\dim A = \dim k[x_1, \dots, x_n]$, and

$$\mathrm{GKdim}\, A = \mathrm{GKdim}\, k[x_1, \dots, x_n] = n.$$

<u>Pf</u>. • Since $A/_{k[x_1, \dots, x_n]}$ is integral, by the Going-Up theorem $\dim A = \dim k[x_1, \dots, x_n]$.

• Since $A/_{k[x_1, \dots, x_n]}$ is integral and A is a f.g. k-alg.,

A is a f.g. $k[x_1, \dots, x_n]$-mod. Hence by a HW assignment

$\mathrm{GKdim}\, A = \mathrm{GKdim}\, k[x_1, \dots, x_n] = n$. ■

Lecture 20: Zero dimensional Noetherian rings

Monday, May 14, 2018 8:50 AM

Next we will study Krull dim. of a ring in more depth. Let's start with a Noetherian ring of dimension 0.

<u>Lemma</u>. Suppose A is Noetherian. Then

$$\dim A = 0 \iff \exists\, \mathfrak{m}_1, \ldots, \mathfrak{m}_n \in \operatorname{Max} A,\ \mathfrak{m}_1 \cdots \mathfrak{m}_n = 0.$$

Moreover, in the above case $\operatorname{Spec} A = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$.

<u>Pf.</u> ($\Leftarrow$) $\dim A = 0 \iff \operatorname{Spec} A = \operatorname{Max} A$.

. $\forall \mathfrak{p} \in \operatorname{Spec} A$, $0 \subseteq \mathfrak{p} \Rightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_n \subseteq \mathfrak{p} \Rightarrow \exists i, \mathfrak{m}_i \subseteq \mathfrak{p}$

$$\Rightarrow \exists i,\ \mathfrak{m}_i = \mathfrak{p}.$$

Hence $\operatorname{Spec} A = \operatorname{Max} A = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$.

($\Rightarrow$) A is Noetherian $\Rightarrow$ 0 has a primary decomposition.

Let $0 = \bigcap_{i=1}^n \mathfrak{q}_i$ be a primary decomposition. Since $\dim A = 0$, $\operatorname{Spec} A = \max A$. Hence $\sqrt{\mathfrak{q}_i} =: \mathfrak{m}_i \in \max A$. Since A is Noetherian $\mathfrak{m}_i$ is a f.g. ideal. And so $\exists\, l \in \mathbb{Z}^+$ s.t. $\mathfrak{m}_i^l \subseteq \mathfrak{q}_i$ for any i. Therefore $\mathfrak{m}_1^l \cdot \mathfrak{m}_2^l \cdots \mathfrak{m}_n^l \subseteq \bigcap_{i=1}^n \mathfrak{q}_i = 0$; and claim follows. ■

Lecture 20: Composition series

Friday, May 18, 2018 8:16 AM

Corollary. A: Noeth. and $\dim A = 0 \Rightarrow$

$\exists\ 0 = \mathfrak{a}_m \subseteq \mathfrak{a}_{m-1} \subseteq \cdots \subseteq \mathfrak{a}_0 = A$ s.t. $\mathfrak{a}_i \trianglelefteq A$ and $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is a simple A-module.

pf. By lemma, $\exists\ \mathfrak{m}_i \in \operatorname{Max} A$ s.t. $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$. Let $\overline{\mathfrak{a}}_0 = A$ and $\overline{\mathfrak{a}}_i := \mathfrak{m}_1 \cdots \mathfrak{m}_i$. Then $\overline{\mathfrak{a}}_i/\overline{\mathfrak{a}}_{i+1} = \overline{\mathfrak{a}}_i/\mathfrak{m}_{i+1}\overline{\mathfrak{a}}_i$ is a vector space over $A/\mathfrak{m}_{i+1}$; and since A is Noetherian, it is a finite dimensional vector space. Hence, it has a "full flag"; that means $\exists\ \overline{\mathfrak{a}}_i = \mathfrak{a}_{i,1} \supseteq \mathfrak{a}_{i,2} \supseteq \cdots \supseteq \mathfrak{a}_{i,d_i} = \overline{\mathfrak{a}}_{i+1}$ s.t. $\mathfrak{a}_{i,j}/\mathfrak{a}_{i,j+1}$ is a 1-dim. $A/\mathfrak{m}_{i+1}$-vector space; and so a simple A-module; and claim follows. $\blacksquare$

Def. For an A-module M, we say $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ is a composition series if M_i's are submodules and M_i/M_{i-1} is a simple A-mod. for any i.

We studied a similar concept for finite groups. Often modules do not have a composition series.