Lecture 21: Composition series

Friday, May 18, 2018 8:31 AM

<u>Proposition</u>. Suppose M is an A-module and it has a composition series of length n. Then any composition series has length <u>n</u> and any series can be extended to a composition series.

<u>Pf</u>. For a module N, let $\ell(N)$ be the length of a shortest compos. series. (If there is no composition series, we say $\ell(N)=\infty$.)

<u>Step 1.</u> If $N \subsetneq M$ is a proper submodule, then $\ell(N) < \ell(M)$.

<u>Pf of step 1.</u> Suppose $0=M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n$ is a compos. series of length n. Let $N_i := M_i \cap N$. Then

$N_i/N_{i-1} \hookrightarrow M_i/M_{i-1}$ as A-mod. So either $N_i = N_{i-1}$ or $N_i/N_{i-1} \simeq M_i/M_{i-1}$ is a simple A-mod. Hence

$0=N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = N$ give us a composition series of length at most n. If length of this composition series is n, then $\forall i$, $N_i \neq N_{i-1}$; and so $\forall i$, $N_i/N_{i-1} \xrightarrow{\sim} M_i/M_{i-1}$.

And, so by induction on i, $N_i = M_i$; which contradicts $N \neq M$. ∎

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Step 2. If $0 = M'_0 \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_m = M$ is a composition series, then $m=n$.

Pf of step 2. $0 = \ell(M'_0) < \ell(M'_1) < \cdots < \ell(M'_m) = \ell(M)$

$\Rightarrow \ell(M) \geq m$. And so $\ell(M) = m$. ∎

Step 3. Suppose $0 = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = M$ is a chain.

So $0 = \ell(K_0) < \ell(K_1) < \cdots < \ell(K_m) = \ell(M)$, which implies $m \leq \ell(M)$.

If $\{K_i\}$ is not a compos. series, then it can be enlarged; but we cannot add more that $\ell(M)$ many modules; and at that point, we should get a composition series. ∎

Def. A module is called Artinian if it satisfies the descending chain condition:

if $M_1 \supseteq M_2 \supseteq \cdots$ is a (descending) chain of submod.,

then $M_n = M_{n+1} = \cdots$ for some $n \in \mathbb{Z}^+$.

Similar to the Noetherian case we have:

Lemma. M is Artinian $\Leftrightarrow$ any $\emptyset \neq \Sigma$ consisting of submod of M has a minimal element.

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Lemma. Suppose M is Artinian, and $N \subseteq M$ is a submodule. Then N and M/N are Artinian.

Lemma. If $M_1, \ldots, M_n$ are Artinian, then $M_1 \oplus \cdots \oplus M_n$ is Artinian.

Lemma. Suppose F is a field, and V is an F-vector space. Then V is Noetherian $\Leftrightarrow$ $\dim V < \infty$ $\Leftrightarrow$ V is Artinian.

Cor. $\ell(M) < \infty \Leftrightarrow M$ is Artinian and Noetherian.

Pf. ($\Rightarrow$) Any chain has length $< \ell(M)$. And so M is Artinian and Noetherian.

($\Leftarrow$) Inductively we define M_i's such that

(0) $M_0 = 0$ (1) M_i/M_{i-1} is a simple A-mod.

For any i, let $\Sigma_i = \{ N \mid M_{i-1} \subsetneq N \underset{\text{submod.}}{\subseteq} M \}$. Then Σ_i has a minimal element; let M_i be a minimal element of Σ_i.

Since M is Noetherian, for some n, $M_n = M$; and so $\ell(M) < \infty$.

To see the last claim, we notice that

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if M_i/M_{i-1} is not a simple A-mod, then $\exists\, 0 \neq N'/M_{i-1} \subsetneq M_i/M_{i-1}$ which is a submod. And so $N' \in \Sigma_i$ and $N' \subsetneq M_i$ which contradicts the minimality of M_i.

<u>Lemma.</u> Suppose $\exists\, \mathfrak{m}_i \in \operatorname{Max} A$ s.t. $\mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \cdots \cdot \mathfrak{m}_n = 0$. Then

A is Noetherian $\Leftrightarrow$ A is Artinian.

<u>Pf.</u> ($\Rightarrow$) We proved in the previous lecture that A has a compos. series $\Rightarrow$ A is Artinian.

($\Leftarrow$) Let $\mathfrak{a}_i = \mathfrak{m}_1 \cdot \cdots \cdot \mathfrak{m}_i$; then $\mathfrak{a}_i/\mathfrak{a}_{i+1} = \mathfrak{a}_i/\mathfrak{m}_{i+1}\cdot\mathfrak{a}_i$ is an $A/\mathfrak{m}_{i+1}$-vector space. Since A is Artinian, $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is an Artinian vector space. Hence it is finite dim.. Hence $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ has a full flag. This implies $\ell(A) < \infty$; and so A is Noetherian. $\blacksquare$

In the next lecture we will prove that

A is Artinian $\Longleftrightarrow$ A is Noetherian and $\dim A = 0$.