We were in the middle of proof of Krull's PIT.

Krull's PIT. Suppose $A$ is Noetherian, $a \notin A^*$, and $\mathfrak{p}$ is a minimal prime ideal of $\langle a \rangle$. Then $\text{ht}(\mathfrak{p}) \leq 1$.

**Pf.** We have made a few reductions, and showed that in addition we can and will assume that $A$ is a local integral domain with maximal ideal $\mathfrak{p}$. And assumed to the contrary that $\mathfrak{p} \nsubseteq \mathfrak{p}_1$ is an intermediate prime. We would like to show: $\text{ht}(\mathfrak{p}_1) = 0$

$$\text{ht}(\mathfrak{p}_1) = 0 \iff \dim A_{\mathfrak{p}_1} = 0 \iff A_{\mathfrak{p}_1} \text{ is Artinian} \iff \exists n, \mathfrak{p}_1^n \mathfrak{p}_1 = \mathfrak{p}_1^{n+1} \mathfrak{p}_1$$

Let $\mathfrak{p}_1^{(k)} := \mathfrak{p}_1^{k} \mathfrak{p}_1 \cap A$. Since $\mathfrak{p}_1 \mathfrak{p}_1$ is maximal, $\mathfrak{p}_1^{(k)} \mathfrak{p}_1$ is $\mathfrak{p}_1 \mathfrak{p}_1$-primary. Hence $\mathfrak{p}_1^{(k)}$ is $\mathfrak{p}_1$-primary, and $\mathfrak{p}_1^{(k)} \supsetneq \mathfrak{p}_1^{(k+1)} \supsetneq \ldots$

Recall that $V(\langle a \rangle) = \{ x \in A : a \mid x \}$; and so $\dim A_{\langle a \rangle} = 0$, which implies $A_{\langle a \rangle}$ is Artinian. Hence $\exists n, \mathfrak{p}_1^{(m)} + \langle a \rangle = \mathfrak{p}_1^{(m+1)} + \langle a \rangle$.

So for any $x_n \in \mathfrak{p}_1^{(m)}$, $\exists x_{n+1} \in \mathfrak{p}_1^{(m+1)}$ and $u_n \in A$ s.t. $x_n = x_{n+1} + u_n a$.

\[
\Rightarrow u_n a \in \mathfrak{p}_1^{(m)} \quad \Rightarrow \quad u_n \in \mathfrak{p}_1^{(m)} \quad \Rightarrow \quad x_n \in \mathfrak{p}_1^{(m+1)} + \mathfrak{p}_1^{(m)} a \subseteq \mathfrak{p}_1^{(m)}
\]

$V(\langle a \rangle) = \{ x \in A : a \mid x \} \Rightarrow a \notin \mathfrak{p}_1^{(m)} \Rightarrow \mathfrak{p}_1^{(m)} = \mathfrak{p}_1^{(m+1)} + a \mathfrak{p}_1^{(m)}$. 


Lecture 27: Converse of Krull's height theorem

Wednesday, June 6, 2018 8:16 AM

As \( a_1 \in \mathfrak{p} \setminus \mathfrak{J}(A) \), by Nakayama's lemma \( \mathfrak{p}_1^n = \mathfrak{p}_1^{n+1} \); and so

\[ \mathfrak{p}_1^n \mathfrak{A}_{\mathfrak{p}_1} = \mathfrak{p}_1^{n+1} \mathfrak{A}_{\mathfrak{p}_1}, \]

which implies \( \mathfrak{p}_1^n \mathfrak{A}_{\mathfrak{p}_1} = \mathfrak{p}_1^{n+1} \mathfrak{A}_{\mathfrak{p}_1} \); and claim follows. \( \blacksquare \)

**Theorem.** Suppose \( A \) is Noetherian, \( \mathfrak{p} \in \text{Spec } A \), and \( \text{ht}(\mathfrak{p}) = d \).

Then \( \exists \mathcal{Q} = \langle a_1, \ldots, a_d \rangle \) st. \( \mathfrak{p} \) is a minimal prime in \( V(\mathcal{Q}) \).

**Proof.** Let \( \mathfrak{p}_0 \not\subsetneq \mathfrak{p}_1 \not\subsetneq \cdots \not\subsetneq \mathfrak{p}_d =: \mathfrak{p} \) be a chain of prime ideals.

Then \( \text{ht}(\mathfrak{p}_i) = i \) as \( \text{ht}(\mathfrak{p}_d) = \text{ht}(\mathfrak{p}) = d \). We prove \( \exists a_1, \ldots, a_d \) st.

1. \( \text{ht}(\langle a_1, \ldots, a_k \rangle) = k \)

2. \( \mathfrak{p}_k \) is a minimal prime in \( V(\langle a_1, \ldots, a_k \rangle) \).

And we prove this by induction on \( d \).

**Base of induction.** Since \( \text{ht} \mathfrak{p}_1 = 1 \), \( \mathfrak{p}_1 \not\subsetneq U \mathfrak{p}' \)

(as otherwise \( \mathfrak{p}_1 \subsetneq \mathfrak{p}' \) for some \( \mathfrak{p}' \) with \( \text{ht} \mathfrak{p}' = 0 \)). Let \( a_1 \mathfrak{p}_1 \setminus U \mathfrak{p}' \)

Since \( a_1 \) is not in a minimal prime ideal, \( \text{ht}(\langle a_1 \rangle) \geq 1 \); and by Krull's PIT, we deduce \( \text{ht}(\langle a_1 \rangle) = 1 \); and so \( \mathfrak{p}_1 \) is a minimal prime in \( V(\langle a_1 \rangle) \).
Induction Step. Suppose \( a_1, ..., a_{d-1} \) satisfy the mentioned conditions. Suppose \( p'_1, ..., p'_m \) are minimal prime in \( V(<a_1, ..., a_{d-1}>) \).

By Krull's height theorem \( \text{ht}(p'_i) \leq d-1 \) and by the induction hypothesis \( \text{ht}(<a_1, ..., a_{d-1}>) = d-1 \) which means \( \min \text{ht}(p'_i) = d-1 \); and so \( \text{ht} p'_i = d-1 \) for any \( i \). Since \( \text{ht} p_d = d \), we deduce that \( p_d \notin \bigcup_{i=1}^m p'_i \). Let \( a_d \in p_d \setminus \bigcup_{i=1}^m p'_i \).

Suppose \( p' \) is a minimal prime of \( <a_1, ..., a_d> \). Hence \( <a_1, ..., a_{d-1}> \subseteq p' \); and \( \exists i \), \( p'_i \subseteq p' \). As \( a_d \in p' \setminus p'_i \), \( p'_i \nsubseteq p' \). Since \( \text{ht} p'_i = d-1 \), \( \text{ht} p' \geq d \). And by Krull's HT \( \text{ht} p' \leq d \); and these imply \( \text{ht} p' = d \).

As \( p_d \in V(<a_1, ..., a_d>) \) and \( \text{ht} p_d = d \), we deduce that \( p_d \) is a minimal prime in \( V(<a_1, ..., a_d>) \). ■
Theorem. Suppose $A$ is Noetherian, and $\text{Max } A = \mathfrak{a} \mathfrak{b} \mathfrak{c}$. Then

$$\dim A = \min_{\mathfrak{q}: \mathfrak{a} \mathfrak{b} \mathfrak{c}-\text{primary}} d(\mathfrak{q})$$

where $d(\mathfrak{q})$ is the minimum number of generators of $\mathfrak{q}$.

\textbf{Proof.} If $\mathfrak{q}$ is $\mathfrak{a} \mathfrak{b} \mathfrak{c}$-primary, then $\sqrt{\mathfrak{q}} = \mathfrak{a} \mathfrak{b} \mathfrak{c}$; and $V(\mathfrak{q}) = \frac{\mathfrak{a} \mathfrak{b} \mathfrak{c}}{\sqrt{\mathfrak{q}}}$. And so by Krull's HT, $\text{ht}(\mathfrak{a} \mathfrak{b} \mathfrak{c}) \leq d(\mathfrak{q})$. Hence

$$\dim A = \text{ht}(\mathfrak{a} \mathfrak{b} \mathfrak{c}) \leq \min_{\mathfrak{q}: \mathfrak{a} \mathfrak{b} \mathfrak{c}-\text{primary}} d(\mathfrak{q}).$$

If $\text{ht}(\mathfrak{a} \mathfrak{b} \mathfrak{c}) = d$, then by the previous theorem $\exists \mathfrak{a}_1, \ldots, \mathfrak{a}_d$ s.t.

$\mathfrak{a} \mathfrak{b} \mathfrak{c}$ is a minimal prime in $V(\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_d \rangle)$. Hence

$V(\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_d \rangle) = \frac{\mathfrak{a} \mathfrak{b} \mathfrak{c}}{\sqrt{\mathfrak{q}_0}}$, which implies $\sqrt{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_d \rangle} = \mathfrak{a} \mathfrak{b} \mathfrak{c}$; and so

$\mathfrak{q}_0 = \langle \mathfrak{a}_1, \ldots, \mathfrak{a}_d \rangle$ is $\mathfrak{a} \mathfrak{b} \mathfrak{c}$-primary. Hence

$$\dim A = \text{ht } \mathfrak{a} \mathfrak{b} \mathfrak{c} = d \geq d(\mathfrak{q}_0) \geq \min_{\mathfrak{q}: \mathfrak{a} \mathfrak{b} \mathfrak{c}-\text{primary}} d(\mathfrak{q});$$

and claim follows. $\blacksquare$

Corollary. Suppose $A$ is Noetherian and $\text{Max } A = \frac{\mathfrak{a} \mathfrak{b} \mathfrak{c}}{\sqrt{\mathfrak{q}}}$. Then

$$\dim A \leq \dim_{A_{\mathfrak{a} \mathfrak{b} \mathfrak{c}}} \frac{\mathfrak{a} \mathfrak{b} \mathfrak{c}}{\mathfrak{q}}.$$

\textbf{Proof.} By Nakayama's lemma $\dim_{A_{\mathfrak{a} \mathfrak{b} \mathfrak{c}}} \frac{\mathfrak{a} \mathfrak{b} \mathfrak{c}}{\mathfrak{q}} = d(\mathfrak{a} \mathfrak{b} \mathfrak{c})$; and claim follows. $\blacksquare$
A local Noetherian ring $A$ with a maximal ideal $\mathfrak{m}$ is called regular if $\dim A = \dim_{A_{\mathfrak{m}}} \mathfrak{m}/\mathfrak{m}^2$.

For $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, the ring of poly. restricted to $X = X(f_1, \ldots, f_r)$ is isomorphic to $A := k[x_1, \ldots, x_n] / \sqrt{\langle f_1, \ldots, f_r \rangle}$.

Suppose $p \in X(f_1, \ldots, f_r)$; and let's consider all the rational functions that are defined at $p$ and restrict them to $X$.

Assuming that $A$ is an integral domain, this ring can be naturally identified with $A_{\mathfrak{m}_p}$; and it is a local Noetherian ring. The tangent plane of $X$ at $p$ can be identified with the dual of $\tilde{\mathfrak{m}}_p / \tilde{\mathfrak{m}}_p^2$ where $\tilde{\mathfrak{m}}_p := \mathfrak{m}_p A_{\mathfrak{m}_p}$; and the regularity assumption $\dim A = \dim_{A_{\mathfrak{m}_p}} \tilde{\mathfrak{m}}_p / \tilde{\mathfrak{m}}_p^2$ is the same as saying that $X$ does not have a singularity at $p$. 