Theorem. Suppose $A$ is a local Noetherian ring and $a \notin D(A) \cup A^x$. Then $\dim A/\langle a \rangle = \dim A - 1$.

**Proof.** Suppose $\text{Max } A = \mathfrak{m} \mathfrak{m} \mathfrak{m}$. Then $\dim A = \text{ht } \mathfrak{m} < \infty$ by Krull’s HT.

Similarly $\dim A/\langle a \rangle = d < \infty$, and let

$\mathfrak{p}_0/\langle a \rangle \neq \mathfrak{p}_1/\langle a \rangle \neq \ldots \neq \mathfrak{p}_d/\langle a \rangle = \mathfrak{m}/\langle a \rangle$

be a saturated chain of prime ideals. By the previous theorem $\exists \overline{a}_1, \ldots, \overline{a}_d$ s.t. $\mathfrak{m}/\langle a \rangle$ is a minimal prime of $V(\langle \overline{a}_1, \ldots, \overline{a}_d \rangle)$.

Since $\text{ht } (\mathfrak{p}_0/\langle a \rangle) = 0$, $\mathfrak{p}_0$ is a minimal prime in $V(\langle a \rangle)$. And so $\text{ht } \mathfrak{p}_0 \leq 1$ by Krull’s PIT. Since $a$ is not a zero-divisor $\text{ht } \mathfrak{p}_0 \neq 0$; and so $\text{ht } \mathfrak{p}_0 = 1$. Let $\mathfrak{p}_{d-1} \not\subseteq \mathfrak{p}_0$. This implies $\text{ht } \mathfrak{m} \geq d + 1$. As $\mathfrak{m}$ is a minimal prime of $V(\langle a, a_1, \ldots, a_d \rangle)$, by Krull’s HT, $\text{ht } \mathfrak{m} \leq d + 1$. Hence $\dim A = \text{ht } \mathfrak{m} = d + 1$.

Def. Suppose $A$ is a local Noetherian ring and $\text{Max } A = \mathfrak{m} \mathfrak{m} \mathfrak{m}$. $(x_1, \ldots, x_r)$ is called an $A$-regular sequence if $x_i \in \mathfrak{m}$ and for any $i$, $x_i \notin D(A/\langle x_1, \ldots, x_{i-1} \rangle)$. 

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Lecture 28: A-regular sequence

Friday, June 8, 2018  2:01 AM

Proposition Suppose $A$ is a local Noetherian ring and $(x_1, \ldots, x_r)$ is an $A$-regular sequence. Then, for any $s \leq r$,

$$\dim \left( A/\langle x_1, \ldots, x_s \rangle \right) = \dim A - s.$$ 

In particular, $r \leq \dim A$.

Proof. We prove this by induction on $s$.

$$x_{s+1} \notin D(A/\langle x_1, \ldots, x_s \rangle) \cup (A/\langle x_1, \ldots, x_s \rangle)^x \Rightarrow$$

$$\dim \left( A/\langle x_1, \ldots, x_{s+1} \rangle \right) = \dim \left( A/\langle x_1, \ldots, x_s \rangle \right) - 1 = \dim A - s - 1.$$ 

Definition. Depth of a local (Noetherian) ring $A$ is the maximum length of an $A$-regular sequence.

Corollary. A local Noetherian $\Rightarrow \text{depth}(A) \leq \dim A$.

Does any local Noetherian ring have an $A$-regular sequence of length $\dim A$? No.

$A := \left( \mathbb{k}[x,y]/\langle x^2, xy \rangle, 0 = \langle \overline{x}, \overline{y} \rangle^2 \cap \langle \overline{x} \rangle, \text{and} \right.$

$D(A) = \langle \overline{x}, \overline{y} \rangle \Rightarrow \text{depth}(A) = 0; \text{ and } \dim A = 1 \text{ as } \langle \overline{x} \rangle \not\subseteq \langle \overline{x}, \overline{y} \rangle.$
Def. A local Noetherian ring $A$ is called Cohen-Macaulay if

\[ \text{depth}(A) = \text{dim}(A) \]

Theorem. Suppose $A$ is a local Noetherian ring and any ideal $\mathfrak{a} \subseteq A$ is unmixed; that means any $\mathfrak{p} \in \text{Ass}(\mathfrak{a})$ is minimal in $\text{Ass}(\mathfrak{a})$. Then $A$ is CM.

Proof. By induction on $r$, we show there is an $A$-regular sequence $(x_1, \ldots, x_r)$ if $r \leq \text{dim} A = d$.

Base. If $d = 0$, we are done. Suppose $d \geq 1$. Claim. $\mathfrak{m} \not\in D(A)$,

where $\text{Max} A = \mathfrak{m} \cup \mathfrak{m} \mathfrak{p}$. If $\mathfrak{m} \subseteq D(A) = \bigcup \mathfrak{p}$, then $\mathfrak{m} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}(\mathfrak{a})$; and so $\mathfrak{m} \in \text{Ass}(\mathfrak{a})$; Since $\mathfrak{a}$ is unmixed, $\mathfrak{m}$ is a minimal prime. Hence $\text{dim} A = 0$; this is a contradiction.

Induction step. If $r = d$, we are done. Suppose $r < d$. Then

\[ \text{dim} \ A/\langle x_1, \ldots, x_r \rangle = d - r > 0 \] by the previous proposition. Hence as $\langle x_1, \ldots, x_r \rangle$ is unmixed, $\mathfrak{m} \not\in \text{Ass}(\langle x_1, \ldots, x_r \rangle)$. Therefore $\mathfrak{m}/\langle x_1, \ldots, x_r \rangle \not\in D(A/\langle x_1, \ldots, x_r \rangle)$, and we can find $x_{r+1}$. \[ \Box \]
Lecture 28: Finitely generated algebras

Remark. We only need to assume any ideal \( \mathfrak{a} \) with \( d(\mathfrak{a}) \leq \dim(A/\mathfrak{a}) \) is unmixed. And in fact converse of this statement is correct as well.

Recall. For \( \mathfrak{p} \in \text{Spec } A \), \( \text{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \max \) length of chain of primes that contain \( \mathfrak{p} \)

And so \( \text{ht} \mathfrak{p} + \dim A/\mathfrak{p} \leq \dim A \). Equality does not hold in general.

Ex. Let \( \mathfrak{a} = \langle x, y \rangle \cap \langle x - 1 \rangle \triangleleft k[x, y] \), and \( A = k[x, y]/\mathfrak{a} \).

Then \( \text{Ass}(\mathfrak{a}) = \{ \langle x, y \rangle, \langle x - 1 \rangle \} \). Hence \( \text{ht}(\mathfrak{p}) = 0 \), \( \text{ht}(\mathfrak{p}') = 1 \);

\( A/\mathfrak{p} \cong k \Rightarrow \dim A/\mathfrak{p} = 0 \). And \( A/\mathfrak{p}' \cong k[x, y]/\langle x - 1 \rangle \cong k[y] \)

\( \Rightarrow \dim A/\mathfrak{p}' = 1 \).

Hence \( \text{ht} \mathfrak{p} + \dim A/\mathfrak{p} = 0 < \text{ht} \mathfrak{p}' + \dim A/\mathfrak{p}' = 1 = \dim A \).

(For any Noetherian ring \( A \), \( \dim A = \max_{\mathfrak{p} \in \text{Ass}(A)} \text{ht} \mathfrak{p} + \dim A/\mathfrak{p} \) as any maximal chain of primes contains a minimal prime.)

Theorem. Suppose \( k \) is a field, and \( A \) is a finitely generated \( k \)-algebra, integral domain. Then any maximal chain of prime ideals of \( A \) has length \( \dim A \). In particular, for any \( \mathfrak{p} \in \text{Spec } A \),

\[ \dim A = \text{ht} \mathfrak{p} + \dim A/\mathfrak{p} \] (Next lecture).