ZARISKI TOPOLOGY

Suppose \( A \) is a unital commutative ring and \( X := \text{Spec}(A) \). For \( f \in A \), let \( X_f := X \setminus V(\langle f \rangle) \).

1. Prove that \( X_f = X_{f'} \) if and only if \( \sqrt{\langle f \rangle} = \sqrt{\langle f' \rangle} \).
2. Prove that there is a bijection between \( X_f \) and \( \text{Spec}(S_f^{-1}A) \) where \( S_f := \{1, f, f^2, \ldots \} \). (We consider the spec of the zero ring to be empty.)
3. Prove that \( \{X_f|f \in A\} \) is a basis of open subsets of \( X \).
4. Prove that every open covering of \( X \) has a finite cover; that means if \( X = \bigcup_{i \in I} \mathcal{O}_i \) where \( \mathcal{O}_i \)'s are open in \( X \), then there is a finite subset \( J \) of \( I \) such that \( X = \bigcup_{j \in J} \mathcal{O}_j \).

MCCOY’S RESULT ON FINITE UNION OF IDEALS

Suppose \( a, b_1, b_2, \ldots, b_k \subseteq A \),

\[
 a \subseteq \bigcup_{i=1}^{k} b_i, \quad \text{and} \quad a \not\subseteq \bigcup_{1 \leq i \leq k, i \neq j} b_i
\]

for any \( 1 \leq j \leq k \). Then there is a positive integer \( n \) such that \( a^n \subseteq \bigcap_{i=1}^{k} b_i \).

(Hint. Use strong induction on \( k \); show that \( a \subseteq b_1 \cup b_2 \) implies \( a \subseteq b_i \) for some \( i \). Argue why \( b_i \cap a \)'s also satisfy the above conditions; and so W.L.O.G. we can assume that \( a = \bigcup_{i=1}^{k} b_i \). By the strong induction hypothesis,

\[
 a \subseteq (b_1 + b_2) \cup b_3 \cup \cdots \cup b_k,
\]

and similar inclusions for any pair \( i \neq j \), deduce that \( a^n \subseteq \prod_{1 \leq i < j \leq k} (b_i + b_j) \) (I). Since \( a = \bigcup_{i=1}^{k} b_i \), deduce that \( \bigcap_{i \neq j} b_i = \bigcap_i b_i \) for any \( j \) (II). Show that claim follows from (I) and (II).)

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Suppose $a, b \subseteq A$; then $(a : b) := \{x \in A | xb \subseteq a\}$. Convince yourself that $(a : b)$ is an ideal of $A$. Prove the following properties:

1. $a \subseteq (a : b)$.
2. $(a : b)b \subseteq a$.
3. $((a : b) : c) = (a : bc) = ((a : c) : b)$.
4. $\bigcap_i (a_i : b) = \bigcap_i (a : b_i)$.
5. $(a : \sum_i b_j) \subseteq \bigcap_i (a : b_j)$.

**Boolean ring as an example**

Here you show that, a ring is locally reduced if and only if it is reduced (reduced means it has no non-zero nilpotent elements). Next you show that there are rings that are not integral domain; but they are locally integral domain. You also show being Noetherian is not a local property either.

1. Suppose, for any $p \in \text{Spec}(A)$, $\text{Nil}(A_p) = 0$. Prove that $\text{Nil}(A) = 0$.  
   (**Hint.** Suppose $x \in \text{Nil}(A)$ and consider $\text{ann}(x) := \{a \in A | ax = 0\}$.)

2. Suppose for any $x \in A$, $x^{n(x)} = x$ for some positive integer $n(x)$. Prove that $\text{Spec}(A) = \text{Max}(A)$.

3. A ring $A$ is called a **Boolean ring** if for any $a \in A$, $a^2 = a$.
   (a) Prove that, for any $p \in \text{Spec}(A)$, $A/p \simeq \mathbb{Z}/2\mathbb{Z}$.
   (b) Prove that, for any $p \in \text{Spec}(a)$, $A_p \simeq \mathbb{Z}/2\mathbb{Z}$.  
   (**Hint.** If $a \in p$, then $1 - a \not\in p$ and $\frac{a}{1} = \frac{a(1-a)}{1-a} = 0$.)
   (c) Let $A := P(X)$ be the power set of a non-empty set $X$. For $a_1, a_2 \in A$, let $a_1 + a_2 := a_1 \Delta a_2$ be the symmetric difference of $a_1$ and $a_2$; that means $a_1 \Delta a_2 := (a_1 \cup a_2) \setminus (a_1 \cap a_2)$. And $a_1 \cdot a_2 := a_1 \cap a_2$. Convince yourself that $(A, +, \cdot)$ is a Boolean ring. Prove that $A$ is Noetherian if and only if $X$ is finite.

4. Suppose, for any $p \in \text{Spec} A$, $A_p$ is an integral domain. Is it true that $A$ is an integral domain?

5. Suppose, for any $p \in \text{Spec} A$, $A_p$ is Noetherian. Is it true that $A$ is Noetherian?