MATH200C, HOMEWORK 4

GOLSEFIDY

Examples related to primary ideals

Suppose k is a field and x, y, and z are indeterminants.

(1) Let $A := k[x, y, z]/\langle xy - z^2\rangle$ and $\mathfrak{p} := \langle \overline{x}, \overline{z}\rangle$ (For any polynomial $f \in k[x, y, z]$, $\overline{f} := f + \langle xy - z^2\rangle \in A$.) Prove that $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{p}^2$ is not primary.

(2) Let $\mathfrak{q} := \langle x, y^2\rangle \trianglelefteq k[x, y]$. Prove that $\mathfrak{m} := \sqrt{\mathfrak{q}}$ is a maximal ideal of $k[x, y]$. Deduce that $\mathfrak{q}$ is $\mathfrak{m}$-primary. Prove that $\mathfrak{q} \neq \mathfrak{m}^n$ for any $n \in \mathbb{Z}^+$.

(3) Let $\mathfrak{a} := \langle x^2, xy\rangle \trianglelefteq k[x, y]$. Show that $\mathfrak{a}$ has at least two reduced primary decompositions.

Ring of polynomials and some primary ideals

For $\mathfrak{a} \trianglelefteq A$, let $\mathfrak{a}[x] := \{\sum_{i=0}^m a_i x^i \mid a_i \in \mathfrak{a}, m \in \mathbb{Z}^+\}$. Let $f : A \hookrightarrow A[x], f(a) := a$.

(1) Convince yourself that $\mathfrak{a}^e = \mathfrak{a}[x]$. Show that $\operatorname{Spec} A \overset{e}{\hookrightarrow} \operatorname{Spec}(A[x])$ is a well-defined injection.

(2) Prove that, if $\mathfrak{q}$ is a $\mathfrak{p}$-primary of A, then $\mathfrak{q}^e$ is a $\mathfrak{p}^e$-primary ideal of $A[x]$.

(3) Suppose k is a field. Prove that $0 \subseteq \langle x_1\rangle \subseteq \cdots \subseteq \langle x_1, \ldots, x_n\rangle$ is a chain of prime ideals of $k[x_1, \ldots, x_n]$ and $\langle x_1, \ldots, x_r\rangle^m$ is $\langle x_1, \ldots, x_r\rangle$-primary for any $1 \leq r \leq n$ and positive integer m.

Faithfully flat algebras

Suppose A and B are two unital commutative rings, and $f : A \to B$ is a ring homomorphism. Consider B as an A-module where $a.b := f(a)b$. Suppose B is a flat A-module. Prove that the following statements are equivalent:

(1) For any $\mathfrak{a} \trianglelefteq A$, $\mathfrak{a}^{ec} = \mathfrak{a}$.

(2) $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective.

(3) For any $\mathfrak{m} \in \operatorname{Max}(A)$, $\mathfrak{m}^e \neq B$.

Date: April 2019.

(4) If M is any non-zero A-module, then $M \otimes_A B \neq 0$.
(5) For any A-module M, $\theta : M \to M \otimes_A B, \theta(x) := x \otimes 1$ is injective.

(**Hint.** (1)$\Rightarrow$(2), in class we proved that $\mathfrak{p}$ is in the image of f^* if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.

(2)$\Rightarrow$(3), if $f^*(\mathfrak{p}') = \mathfrak{m}$, then $\mathfrak{m}^e \subseteq \mathfrak{p}'$.

(3)$\Rightarrow$(4), For any $x \in M$, $0 \to Ax \to M$ is exact. Since B is flat, $0 \to Ax \otimes_A B \to M \otimes_A B$ is exact. So to show $M \otimes_A B$ is not zero, it is enough to show $Ax \otimes_A B$ is not zero. Suppose $\mathfrak{a} := \{a \in A |\ ax = 0\}$; then $Ax \simeq A/\mathfrak{a}$ as an A-module. Hence $Ax \otimes_A B \simeq B/\mathfrak{a}^e$ as an A-module. Suppose $\mathfrak{m}$ is a maximal ideal such that $\mathfrak{a} \subseteq \mathfrak{m}$, and deduce the claim.

(4)$\Rightarrow$(5), Suppose $M' := \ker \theta$. Since B is a flat A-module,

$$0 \to M' \otimes_A B \to M \otimes_A B \xrightarrow{g} (M \otimes_A B) \otimes_A B$$

is exact, where $g := \theta \otimes \mathrm{id}_B$. View $M \otimes_A B$ as an B-module and let

$$h : (M \otimes_A B) \otimes_A B \to M \otimes_A B, h(x \otimes b) := xb.$$

Show that h is a well-defined B-module homomorphism. Notice that $g(m \otimes b) = \theta(m) \otimes b = (m \otimes 1) \otimes b$; and so $(h \circ g)(m \otimes b) = (m \otimes 1)b = m \otimes b$. This implies that $h \circ g = \mathrm{id}$. Deduce that g is injective.

(5)$\Rightarrow$(1) Show that $\overline{f} : A/\mathfrak{a} \to B/\mathfrak{a}^e, \overline{f}(a + \mathfrak{a}) := f(a) + \mathfrak{a}^e$ is a well-defined injective ring homomorphism.)

Kummer theory

Suppose F is a field and its characteristic is not 2. Let $a_1, \ldots, a_n \in F^\times$, $H := \langle a_1(F^\times)^2, \ldots, a_n(F^\times)^2 \rangle \leq F^\times/(F^\times)^2$, and $E := F[\sqrt{a_1}, \ldots, \sqrt{a_n}]$.

(1) Prove that E/F is a Galois extension.
(2) Let $G := \mathrm{Gal}(E/F)$. Prove that any non-trivial element of G has order 2. Deduce that $G \simeq (\mathbb{Z}/2\mathbb{Z})^m$ for some $m \in \mathbb{Z}$.
(3) Prove that $H \simeq (\mathbb{Z}/2\mathbb{Z})^k$ for some $k \in \mathbb{Z}$.

(4) Let $T : G \times H \to \{\pm 1\} \simeq \mathbb{Z}/2\mathbb{Z}, T(\sigma, a(F^\times)^2) := \sigma(\sqrt{a})/\sqrt{a}$. Prove that T is a non-degenerate bilinear form; that means

$$\begin{aligned} T(\sigma\sigma', \overline{a}) =& T(\sigma, \overline{a})T(\sigma', \overline{a}), \\ T(\sigma, \overline{a}\overline{a}') =& T(\sigma, \overline{a})T(\sigma, \overline{a}'), \\ (\forall \sigma \in G, T(\sigma, \overline{a}_0) = 1) \Rightarrow& \ \overline{a}_0 = \overline{1}, \\ (\forall \overline{a} \in H, T(\sigma_0, \overline{a}) = 1) \Rightarrow& \ \sigma_0 = \mathrm{id}_E. \end{aligned}$$

(5) Prove that $\mathrm{Gal}(F[\sqrt{a_1}, \ldots, \sqrt{a_n}]/F) \simeq \langle a_1(F^\times)^2, \ldots, a_n(F^\times)^2 \rangle$.