MATH200C, HOMEWORK 6

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Valuation rings and valuations.

(1) Suppose G is a totally ordered abelian group, and F is a field. A valuation of F is $v : F \to G \cup \{\infty\}$ with the following properties:
   (a) $v(a) = \infty \Leftrightarrow a = 0$.
   (b) $\forall g \in G, g < \infty, g + \infty := \infty, \infty + \infty := \infty$.
   (c) $\forall x, y \in F, v(xy) = v(x) + v(y)$.
   (d) $\forall x, y \in F, v(x + y) \geq \min\{v(x), v(y)\}$.

   Let $\mathscr{O}_v := \{x \in F | v(x) \geq 0\}$ and $\mathfrak{m}_v := \{x \in F | v(x) > 0\}$. Prove that $\mathscr{O}_v$ is a valuation ring and $\mathrm{Max}(\mathscr{O}_v) = \{\mathfrak{m}_v\}$.

(2) Let A be a valuation ring of a field F. Let $G := F^\times / A^\times$. We say $xA^\times \geq yA^\times$ if $xy^{-1} \in A$. Prove that G is a totally ordered group and $v : F \to G \cup \{\infty\}, v(x) := \begin{cases} xA^\times & \text{if } x \in F^\times \\ \infty & \text{otherwise.} \end{cases}$ is a valuation of F, and $\mathscr{O}_v = A$.

Ring of integers of a number field

Suppose $k/\mathbb{Q}$ is a finite field extension (we say k is a number field), and $\mathscr{O}_k$ is the integral closure of $\mathbb{Z}$ in k. In class we proved that $\dim \mathscr{O}_k = 1$ and $\mathscr{O}_k \simeq \mathbb{Z}^{[k:\mathbb{Q}]}$ as an abelian group.

(1) Prove that for any $\mathfrak{m} \in \mathrm{Max}(\mathscr{O}_k)$, $\mathscr{O}_k/\mathfrak{m}$ is a finite field.
(2) Prove that if $\mathfrak{q}$ is a non-zero primary ideal of $\mathscr{O}_k$, then $\mathscr{O}_k/\mathfrak{q}$ is finite.
(3) Prove that if $\mathfrak{a}$ is a non-zero ideal of $\mathscr{O}_k$, then $\mathscr{O}_k/\mathfrak{a}$ is a finite. ($|\mathscr{O}_k/\mathfrak{a}|$ is called the norm of the ideal $\mathfrak{a}$, and it is denoted by $N_{k/\mathbb{Q}}(\mathfrak{a})$.)
(4) Suppose $\mathrm{Embed}_{\mathbb{Q}}(k, \overline{\mathbb{Q}}) = \{\sigma_1, \ldots, \sigma_d\}$, and $\mathscr{O}_k = \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \cdots \oplus \mathbb{Z}a_d$. Prove that $D_k := \det[\sigma_i(a_j)]^2 \in \mathbb{Z}$ (this is called the discriminant of k).
(5) For $a \in \mathscr{O}_k \setminus \{0\}$, prove that $|N_{k/\mathbb{Q}}(a)| = |\mathscr{O}_k / a\mathscr{O}_k|$. (**Hint**. Let $l_a : k \to k, l_a(x) := ax$. Then $N_{k/\mathbb{Q}}(a) = \det l_a$. On the other hand, think about $\mathscr{O}_k$

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as a lattice in $\mathscr{O}_k \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}a_1 \oplus \mathbb{R}a_2 \oplus \cdots \oplus \mathbb{R}a_d$ and consider the $\mathbb{R}$-linear map $T_a := l_a \otimes \mathrm{id}_{\mathbb{R}} : \mathscr{O}_k \otimes_{\mathbb{Z}} \mathbb{R} \to \mathscr{O}_k \otimes_{\mathbb{Z}} \mathbb{R}$ and argue why $\det l_a = \det T_a$. Argue why using volume we have that for $T_a \in M_d(\mathbb{Z})$, $|\det T_a| = [\mathbb{Z}^d : T_a(\mathbb{Z}^d)]$.) (Because of this part we have $N_{k/\mathbb{Q}}(a\mathscr{O}_k) = |N_{k/\mathbb{Q}}(a)|$.)

Gelfand-Kirillov dimension

Let k be a field and A be a finitely generated k-algebra. Suppose V is a k-subspace of A such that

(1) $\dim_k V < \infty$,
(2) $1 \in V$,
(3) $A = \bigcup_{i=1}^{\infty} V^{(n)}$ where $V^{(n)} := \{v_1 \cdots v_n | v_i \in V\}$.

Let $D_V(n) := \dim_k V^{(n)}$ and

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log D_V(n)}{\log n}. \tag{1}$$

(1) Prove that the value of (1) is independent of choice of V.
(2) Suppose B/A is a ring extension and A and B are finitely generated k-algebras. Prove that $\mathrm{GKdim}(A) \leq \mathrm{GKdim}(B)$.
(3) Suppose $\mathfrak{a} \trianglelefteq A$. Prove that $\mathrm{GKdim}(A/\mathfrak{a}) \leq \mathrm{GKdim}(A)$.
(4) Prove that $\mathrm{GKdim}(A[x]) = \mathrm{GKdim}(A) + 1$.
(5) Suppose B/A is a ring extension and B is a finitely generated A-module. Prove that $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$.
(6) Using the fact that $\dim k[x_1, \ldots, x_d] = d$, prove that $\mathrm{GKdim}(A) = \dim A$.

(The importance of GKdimension is on the fact that it works equally well in the non-commutative setting; in fact the above (1)-(5) statements hold for non-commutative algebras as well.)