MATH200C, LECTURE 8

GOLSEFIDY

Basics of primary ideals

In the previous lecture we were proving:

Lemma 1. *If* $\mathfrak{m} \in \mathrm{Max}(A)$ *and* $\sqrt{\mathfrak{q}} = \mathfrak{m}$*, then* $\mathfrak{q}$ *is* $\mathfrak{m}$*-primary.*

Proof. Since $\sqrt{\mathfrak{q}} = \bigcap_{\mathfrak{p}\in V(\mathfrak{q})} \mathfrak{p} = \mathfrak{m}$, we have that $\mathfrak{q} \subseteq \mathfrak{p} \Rightarrow \mathfrak{m} \subseteq \mathfrak{p}$. Since $\mathfrak{m}$ is a maximal ideal, we have $V(\mathfrak{q}) = \{\mathfrak{m}\}$.

Suppose $x \notin \mathfrak{q}$ and $xy \in \mathfrak{q}$. Consider

$$(\mathfrak{q} : x) := \{a \in A |\; ax \in \mathfrak{q}\}.$$

Then one can check that $(\mathfrak{q} : x)$ is an ideal of A, $\mathfrak{q} \subseteq (\mathfrak{q} : x)$ (alternatively $(\mathfrak{q} : x)|\mathfrak{q}$), and $y \in (\mathfrak{q} : x)$. Hence $V(\mathfrak{q} : x) \subseteq V(\mathfrak{q}) = \{\mathfrak{m}\}$; and so either $(\mathfrak{q} : x) = A$ or $V((\mathfrak{q} : x)) = \{\mathfrak{m}\}$. Since $x \notin \mathfrak{q}$, $1 \notin (\mathfrak{q} : x)$. Thus $V((\mathfrak{q} : x)) = \{\mathfrak{m}\}$, which implies that $y \in (\mathfrak{q} : x) \subseteq \mathfrak{m}$. This implies that $\mathfrak{q}$ is primary. □

As it has been mentioned earlier, primary ideals are supposed to play the role of powers of primes. The next lemma shows that when A is a PID these two concepts are equivalent.

Lemma 2. *Suppose A is a PID. Then $\mathfrak{q}$ is a non-zero primary ideal of A if and only if there is a prime element p of A and positive integer n such that $\mathfrak{q} = \langle p^n \rangle$.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{p} := \sqrt{\mathfrak{q}}$. So $\mathfrak{p} \in \mathrm{Spec}(A) = \{0\} \cup \mathrm{Max}(A)$. Notice that $\mathfrak{p} = 0$ if and only if $\mathfrak{q}$. If $\mathfrak{p} \neq 0$, then there is an irreducible element $p \in A$ such that $\mathfrak{p} = \langle p \rangle$; and in a PID an element is irreducible if and only if it is prime. Suppose $\mathfrak{q} = \langle a \rangle$. Since $\mathfrak{p}$ is the smallest prime divisor of $\mathfrak{q}$, we have that, if ℓ is prime in A and $\ell | a$, then $p|\ell$; this means p is the only prime factor of a. Hence there is positive integer n such that $\langle a \rangle = \langle p^n \rangle$.

($\Leftarrow$) $\sqrt{\langle p^n \rangle} = \langle p \rangle \in \mathrm{Max}(A)$. Hence by the previous lemma, $\langle p^n \rangle$ is primary. □

As we have seen it the proof of Lemma 1, it is instrumental to understand $(\mathfrak{q}:x)$ to have a better understanding of $\mathfrak{q}$.

Lemma 3. *Suppose $\mathfrak{q}$ is $\mathfrak{p}$-primary. Then*

$$(\mathfrak{q}:x)=\begin{cases} A & \text{if } x\in\mathfrak{q},\\ \mathfrak{q} & \text{if } x\not\in\mathfrak{p},\\ \mathfrak{p}\text{-primary} & \text{if } x\not\in\mathfrak{q}. \end{cases}$$

Proof. If $x\in\mathfrak{q}$, then it is clear that $(\mathfrak{q}:x)=A$.

Suppose $x \not in \mathfrak{p}$; then $y\in(\mathfrak{q}:x)$ implies that $xy\in\mathfrak{q}$. Since $\mathfrak{q}$ is primary, $xy\in\mathfrak{q}$ and $x\not\in\mathfrak{q}$ imply that $y\in\sqrt{\mathfrak{q}}=\mathfrak{p}$; this is a contradiction. Therefore $(\mathfrak{q}:x)\subseteq\mathfrak{q}$. And for any ideal $\mathfrak{q}$ and any element x, we have $(\mathfrak{q}:x)\supseteq\mathfrak{q}$.

Suppose $x\not\in\mathfrak{q}$. First we show that $\sqrt{(\mathfrak{q}:x)}=\mathfrak{p}$. Suppose $y\in\sqrt{(\mathfrak{q}:x)}$. Then for some positive integer n, $y^n x\in\mathfrak{q}$. Since $\mathfrak{q}$ is a primary ideal, $x \not\mathfrak{q}$ and $xy^n\in\mathfrak{q}$ imply that, for some positive integer m, $(y^n)^m\in\mathfrak{q}$. This means $y\in\sqrt{\mathfrak{q}}=\mathfrak{p}$. Hence $\sqrt{(\mathfrak{q}:x)}\subseteq\sqrt{\mathfrak{q}}$. We always have $\sqrt{(\mathfrak{q}:x)}\supseteq\sqrt{\mathfrak{q}}$; and so $\sqrt{(\mathfrak{q}:x)}=\mathfrak{p}$.

Suppose $yz\in(\mathfrak{q}:x)$ and $y\not\in\sqrt{(\mathfrak{q}:x)}=\mathfrak{p}$. Hence $(xz)y\in\mathfrak{q}$ and $y\not\in\sqrt{\mathfrak{q}}$. As $\mathfrak{q}$ is primary, we can deduce that $xz\in\mathfrak{q}$; this means $z\in(\mathfrak{q}:x)$. Therefore $(\mathfrak{q}:x)$ is $\mathfrak{p}$-primary. ∎

Primary decomposition

Definition 4. *An ideal $\mathfrak{a}$ is called decomposable if there are finitely many primary ideals $\mathfrak{q}_i$ such that $\mathfrak{a}=\bigcap_{i=1}^n\mathfrak{q}_i$.*

A decomposition $\bigcap_{i=1}^n\mathfrak{q}_i$ is called reduced *if*

(1) for any i, $\mathfrak{q}_i\not\supseteq\bigcap_{j\neq i}\mathfrak{q}_j$,
(2) $\sqrt{\mathfrak{q}_i}\neq\sqrt{\mathfrak{q}_j}$ for $i\neq j$.

Lemma 5. *(1) Suppose $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary; then $\mathfrak{q}\cap\mathfrak{q}'$ is $\mathfrak{p}$-primary.*
(2) A decomposable ideal has a reduced decomposition.

Proof. (1) As $\mathfrak{q}\cap\mathfrak{q}'\subseteq\mathfrak{q}$, $\sqrt{\mathfrak{q}\cap\mathfrak{q}'}\subseteq\sqrt{\mathfrak{q}}=\mathfrak{p}$. If $x\in\mathfrak{p}$, then there are positive integers n,n' such that $x^n\in\mathfrak{q}$ and $x^{n'}\in\mathfrak{q}'$. Hence for any $m\geq\max(n,n')$, $x^m\in\mathfrak{q}\cap\mathfrak{q}'$; and so $x\in\sqrt{\mathfrak{q}\cap\mathfrak{q}'}$. Thus $\sqrt{\mathfrak{q}\cap\mathfrak{q}'}=\mathfrak{p}$. Suppose $xy\in\mathfrak{q}\cap\mathfrak{q}'$ and $x\not\in\sqrt{\mathfrak{q}\cap\mathfrak{q}'}=\mathfrak{p}$. $xy\in\mathfrak{q}$ and $x\not\in\mathfrak{p}=\sqrt{\mathfrak{q}}$ imply that $y\in\mathfrak{q}$; and similarly $xy\in\mathfrak{q}'$ and $x\not\in\mathfrak{p}=\sqrt{\mathfrak{q}'}$ imply that $y\in\mathfrak{q}'$. Thus $y\in\mathfrak{q}\cap\mathfrak{q}'$; and claim follows.

(2) We start with a decomposition $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$. Using part (1), we can make sure that $\sqrt{\mathfrak{q}}_i \neq \sqrt{\mathfrak{q}}_j$ if $i \neq j$. And then we can drop any unnecessary $\mathfrak{q}_i$ if needed, to end up getting a reduced decomposition. $\square$

How much is a primary decomposition unique? In your HW assignment you will see examples of ideals with at least two primary decompositions. That said some parameters of a reduced primary decomposition of an ideal $\mathfrak{a}$ just depends on $\mathfrak{a}$.

Theorem 6. *Suppose $\bigcap_{i=1}^n \mathfrak{q}_i$ is a reduced primary decomposition of $\mathfrak{a}$, and $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. Then*

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \operatorname{Spec}(A) \cap \{\sqrt{(\mathfrak{a} : x)} | \, x \in A\};$$

in particular $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ just depends on $\mathfrak{a}$ and it is independent of the choice of a reduced primary decomposition.

If $\mathfrak{a}$ is decomposable and $\bigcap_{i=1}^n \mathfrak{q}_i$ is a reduced primary decomposition, then $\{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_n}\}$ is called the set of primes associated with $\mathfrak{a}$; and we write $\operatorname{Ass}(\mathfrak{a}) := \{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_n}\}$.

Proof. We make notice of two things:

$$(\bigcap_{i\in I} \mathfrak{b}_i : x) = \bigcap_{i\in I} (\mathfrak{b}_i : x) \text{ and } \sqrt{\bigcap_{i=1}^n \mathfrak{b}_i} = \bigcap_{i=1}^n \sqrt{\mathfrak{b}_i}.$$

Here is their proof:

$$y \in (\bigcap_{i\in I} \mathfrak{b}_i : x) \Leftrightarrow xy \in \bigcap_{i\in I} \mathfrak{b}_i \Leftrightarrow \forall i \in I, xy \in \mathfrak{b}_i \Leftrightarrow \forall i \in I, y \in (\mathfrak{b}_i : x) \Leftrightarrow y \in \bigcap_{i\in I} \mathfrak{b}_i.$$

Since $\bigcap_{i=1}^n \mathfrak{b}_i \subseteq \mathfrak{b}_i$ for any i, $\sqrt{\bigcap_{i=1}^n \mathfrak{b}_i} \subseteq \bigcap_{i=1}^n \sqrt{\mathfrak{b}_i}$; and

$$y \in \bigcap_{i=1}^n \sqrt{\mathfrak{b}_i} \Rightarrow \forall i, y \in \sqrt{\mathfrak{b}_i} \Rightarrow \forall i, \exists n_i \in \mathbb{Z}^+, y^{n_i} \in \mathfrak{b}_i \Rightarrow y^{\max_i(n_i)} \in \bigcap_{i=1}^n \mathfrak{b}_i \Rightarrow y \in \sqrt{\bigcap_{i=1}^n \mathfrak{b}_i}.$$

Hence

$$\sqrt{(\mathfrak{a} : x)} = \sqrt{(\bigcap_{i=1}^n \mathfrak{q}_i : x)} = \bigcap_{i=1}^n \sqrt{(\mathfrak{q}_i : x)}. \tag{1}$$

By the Lemma 3, $\sqrt{(\mathfrak{q}_i : x)} = A$ if $x \in \mathfrak{q}_i$ and $\sqrt{(\mathfrak{q}_i : x)} = \mathfrak{p}_i$ if $x \notin \mathfrak{q}_i$. Hence by (1) we have

$$\sqrt{(\mathfrak{a} : x)} = \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i$$

with the understanding that if $x \in \bigcap_{i=1}^n \mathfrak{q}_i$, then the above intersection is A. Since $\mathfrak{q}_j$'s give us a reduced primary decomposition, there is $x_i \in \bigcap_{j \neq i} \mathfrak{q}_j \setminus \mathfrak{q}_i$. Then, by (1), $\sqrt{(\mathfrak{a} : x_i)} = \mathfrak{p}_i$. This means the RHS is a subset of the LHS in the statement of Theorem.

Suppose $\sqrt{(\mathfrak{a} : x)} =: \mathfrak{p}$ is a prime ideal. Then by (1), $\mathfrak{p} = \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i$. Since $\mathfrak{p}$ is prime, $\bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i \subseteq \mathfrak{p}$ implies that for some i_0, $x_{i_0} \notin \mathfrak{q}_{i_0}$ and $\mathfrak{p}_{i_0} \subseteq \mathfrak{p}$. Since $\mathfrak{p} \subseteq \bigcap_{x_i \notin \mathfrak{q}_i} \mathfrak{p}_i \subseteq \mathfrak{p}_{i_0}$, we have $\mathfrak{p} \subseteq \mathfrak{p}_{i_0}$. Altogether, we have $\mathfrak{p} = \mathfrak{p}_{i_0}$ for some i_0. This implies that the RHS is a subset of the LHS; and claim follows. □

Proposition 7. *Suppose $\mathfrak{a}$ is decomposable. Then*

(1) $\mathrm{Ass}(\mathfrak{a}) \subseteq V(\mathfrak{a})$.
(2) For any $\mathfrak{p} \in V(\mathfrak{a})$, there is $\mathfrak{p}' \in \mathrm{Ass}(\mathfrak{a})$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$.
(3) The set of minimal elements of $\mathrm{Ass}(\mathfrak{a})$ with respect to inclusion is the same as the set of minimal elements of $V(\mathfrak{a})$ with respect to inclusion.

We deduce that, if $\mathfrak{a}$ is decomposable, then $V(\mathfrak{a})$ has only finitely many minimal elements. We will prove later that if A is Noetherian, then any ideal is decomposable. This is similar to how we used a chain condition to prove that any element can be written as a product of irreducible elements in a Noetherian integral domian.

Proof. (1) Suppose $\bigcap_{i=1}^n \mathfrak{q}_i$ is a reduced primary decomposition of $\mathfrak{a}$ and $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. Then $\mathfrak{a} \subseteq \mathfrak{q}_i \subseteq \mathfrak{p}_i$ for any i; and so $\mathfrak{p}_i \in V(\mathfrak{a})$.

(2) For any $\mathfrak{p} \in V(\mathfrak{a})$, we have $\bigcap_{i=1}^n \mathfrak{q}_i \subseteq \mathfrak{p}$. Hence $\sqrt{\bigcap_{i=1}^n \mathfrak{q}_i} \subseteq \sqrt{\mathfrak{p}}$ which implies

$$\bigcap_{i=1}^n \mathfrak{p}_i = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i} \subseteq \mathfrak{p}.$$

Since $\mathfrak{p}$ is prime, we have that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i.

(3) Suppose $\mathfrak{p}$ is a minimal element of $V(\mathfrak{a})$. By (2), there is $\mathfrak{p}' \in \mathrm{Ass}(\mathfrak{a})$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. As $\mathrm{Ass}(\mathfrak{a}) \subseteq V(\mathfrak{a})$ and $\mathfrak{p}$ is minimal in $V(\mathfrak{a})$, $\mathfrak{p}' \subseteq \mathfrak{p}$ implies that $\mathfrak{p} = \mathfrak{p}'$. Since $\mathfrak{p}$ is minimal in $V(\mathfrak{a})$, $\mathrm{Ass}(\mathfrak{a}) \subseteq V(\mathfrak{a})$, and $pfr \in \mathrm{Ass}(A)$, $\mathfrak{p}$ is minimal in $\mathrm{Ass}(A)$. Hence

minimal in $V(\mathfrak{a})$ implies minimal in $\mathrm{Ass}(\mathfrak{a})$.

Suppose $\mathfrak{p}$ is minimal in $\mathrm{Ass}(\mathfrak{a})$. And suppose to the contrary that $\mathfrak{p}$ is not minimal in $V(\mathfrak{a})$. Then there is $\overline{\mathfrak{p}} \in V(\mathfrak{a})$ such that $\overline{\mathfrak{p}} \subsetneq \mathfrak{p}$. By part (2), there is $\mathfrak{p}' \in \mathrm{Ass}(\mathfrak{a})$ such that $\mathfrak{p}' \subseteq \overline{\mathfrak{p}}$. Thus

$$\mathfrak{p}' \subseteq \overline{\mathfrak{p}} \subsetneq \mathfrak{p};$$

but $\mathfrak{p}', \mathfrak{p} \in \mathrm{Ass}(\mathfrak{a})$ and $\mathfrak{p}' \subsetneq \mathfrak{p}$ contradict that $\mathfrak{p}$ is minimal in $\mathrm{Ass}(\mathfrak{a})$. Therefore

minimal in $V(\mathfrak{a})$ implies minimal in $V(\mathfrak{a})$;

and claim follows. ☐

Proposition 8. *Suppose* $\mathfrak{a}$ *is decomposable. Then*

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}(\mathfrak{a})} \mathfrak{p} = \{x \in A | \ (\mathfrak{a} : x) \neq \mathfrak{a}\}.$$

Notice that $\{x \in A | \ (0 : x) \neq 0\} = D(A)$ is the set of zero-divisors of A. And so the above proposition implies $D(A) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(0)} \mathfrak{p}$ if 0 is decomposable.

Proof. Suppose $\bigcap_{i=1}^n \mathfrak{q}_i$ is a reduced primary decomposition of $\mathfrak{a}$. Then for any x, $(\mathfrak{a} : x) = \bigcap_{i=1}^n (\mathfrak{q}_i : x)$. So if $(\mathfrak{a} : x) \neq \mathfrak{a}$, then for some i we have that $(\mathfrak{q}_i : x) \neq \mathfrak{q}_i$. Therefore, by Lemma 3, $x \in \mathfrak{p}_i$. Hence the RHS is a subset of the LHS.

Suppose $x \in \mathfrak{p}_i$ for some i. Then by Theorem 6, there is $y \in A$ such that $\mathfrak{p}_i = \sqrt{(\mathfrak{a} : y)}$. So $x \in \sqrt{(\mathfrak{a} : y)}$; hence for some positive integer n, $x^n \in (\mathfrak{a} : y)$, which implies that $x^n y \in \mathfrak{a}$.

Now suppose to the contrary that $(\mathfrak{a} : x) = \mathfrak{a}$. In this case, we claim that $y \in \mathfrak{a}$. To show this suppose i is the smallest non-negative integer such that $x^i y \in \mathfrak{a}$. If $i = 0$, we get the claim. If $i > 0$, then $x(x^{i-1}y) \in \mathfrak{a}$ implies that $x^{i-1}y \in (\mathfrak{a} : x) = \mathfrak{a}$; and this contradicts the minimality of i. Hence $y \in \mathfrak{a}$, which implies that $(\mathfrak{a} : y) = A$; but this contradicts that $\sqrt{(\mathfrak{a} : y)}$ is prime. Therefore $(\mathfrak{a} : x) \neq \mathfrak{a}$, which means the LHR is a subset of the RHS. ☐