MATH200C, LECTURE 12

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Dimension and integral extension

Lemma 1. *Suppose* $f : A \hookrightarrow B$ *is integral,* $\mathfrak{p} \in \mathrm{Spec}(A)$*, and* $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \in (f^*)^{-1}(\mathfrak{p})$*. Then* $\mathfrak{q}_1 = \mathfrak{q}_2$*.*

Proof. Since $f : A \hookrightarrow B$ is integral, $\overline{f} : A/f^*(\mathfrak{q}_1) \hookrightarrow B/\mathfrak{q}_1$ is integral; and we have $\overline{f}^*(\overline{\mathfrak{q}}_2) = 0$. It is enough to show $\overline{\mathfrak{q}}_2 = 0$. Suppose to the contrary that $\overline{a} \in \overline{\mathfrak{q}}_2 := \mathfrak{q}_2/\mathfrak{q}_1$ is not zero, and $f(x) \in (A/\mathfrak{p})[x]$ is the smallest positive degree monic polynomial that has $\overline{a}$ as a zero; say $f(x) = x^n + \overline{a}_{n-1}x^{n-1} + \cdots + \overline{a}_0$. Then

$$\overline{a}_0 = -\overline{a}(\overline{a}^{n-1} + \overline{a}_{n-1}\overline{a}^{n-2} + \cdots + \overline{a}_1) \in \overline{\mathfrak{q}}_2 \cap \overline{f}(A/\mathfrak{p});$$

and so $\overline{a}_0 \in \overline{f}^*(\overline{\mathfrak{q}}_2) = 0$. This implies that $\overline{a}$ is a zero $x^{n-1} + \overline{a}_{n-1}x^{n-2} + \cdots + \overline{a}_{n-1}$, which contradicts the way we chose f. □

Theorem 2. *Suppose* B/A *is an integral extension. Then* $\dim A = \dim B$*.*

Proof. Suppose $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ is a chain in $\mathrm{Spec}(B)$. Then clearly $f^*(\mathfrak{q}_i) \subseteq f^*(\mathfrak{q}_{i+1})$ for any i; and by the previous lemma, equality cannot hold and so

$$f^*(\mathfrak{q}_0) \subsetneq f^*(\mathfrak{q}_1) \subsetneq \cdots \subsetneq f^*(\mathfrak{q}_n)$$

is a chain in $\mathrm{Spec}(A)$, which implies that $\dim B \le \dim A$.

For any chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in $\mathrm{Spec}(A)$, by the Going-Up Theorem there is a chain

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n$$

in $\mathrm{Spec}(B)$ (such that $f^*(\mathfrak{q}_i) = \mathfrak{p}_i$ for any i). Hence $\dim A \le \dim B$; and claim follows. □

Integral over an ideal

So far we have proved that if $f : A \hookrightarrow B$ is integral, then f^* is onto and closed, and its fibers have dimension 0. Next we want to show that under certain additional conditions, f^* is also open, and get a better understanding of its fibers. To this end, we start with a technical lemma, and we will see its importance later.

Suppose B/A is a ring extension and $\mathfrak{a} \trianglelefteq A$. We say $b \in B$ is integral over A if there are $a_i \in \mathfrak{a}$ such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0.$$

Lemma 3. *Suppose B/A is a ring extension and C is the integral closure of A in B. Suppose $\mathfrak{a} \trianglelefteq A$. Then*

$$b \in B \text{ is integral over } \mathfrak{a} \Leftrightarrow b \in \sqrt{\mathfrak{a}^e}$$

where $\mathfrak{a}^e$ is the extension of $\mathfrak{a}$ in C; in particular if b_1 and b_2 are integral over $\mathfrak{a}$, then so are $b_1 \pm b_2$ and $b_1 b_2$.

Proof. ($\Rightarrow$) Suppose b is integral over $\mathfrak{a}$; that means there are $a_i \in \mathfrak{a}$ such that $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$. Hence $b \in C$ and

$$b^n = -(a_{n-1}b^{n-1} + \cdots + a_0) \in \mathfrak{a}^e,$$

which implies that $b \in \sqrt{\mathfrak{a}^e}$.

($\Leftarrow$) Suppose $b \in \sqrt{\mathfrak{a}^e}$; then there are $c_i \in C$, $a_i \in \mathfrak{a}$, and $n \in \mathbb{Z}^+$ such that

$$b^n = c_1 a_1 + \cdots + c_m a_m. \tag{1}$$

Let $\overline{C} := A[c_1, \ldots, c_m]$. Since c_i's are integral over A, $\overline{C}$ is a finitely generated A-module. By (1) we have

$$l_{b^n}(\overline{C}) \subseteq \mathfrak{a}\overline{C},$$

where $\phi := l_{b^n} : \overline{C} \to \overline{C}, l_{b^n}(x) := b^n x$ is an A-module homomorphism. Therefore by a result that we proved earlier (which was used to show Nakayama's lemma) we have

$$\phi^k + a'_{k-1}\phi^{k-1} + \cdots + a'_0 = 0$$

in $\mathrm{End}_A(\overline{C})$ for some $a'_i \in \mathfrak{a}$. This implies that

$$b^{nk} + a'_{k-1}b^{n(k-1)} + \cdots + a'_0 = 0$$

for some $a'_i \in \mathfrak{a}$; and claim follows. $\square$

Minimal polynomial revisited

Lemma 4. *Suppose B/A is an integral extension, B is an integral domain, A is integrally closed, and F is the field of fractions of A. Suppose $\mathfrak{a} \trianglelefteq A$ and $b \in B$ is integral over $\mathfrak{a}$. Let*

$$m_{b,F}(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_0 \in F[x]$$

be the minimal polynomial of b over F. Then $c_i \in \sqrt{\mathfrak{a}}$ for any i.

Proof. Since b is integral over $\mathfrak{a}$, there is $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$ such that $a_i \in \mathfrak{a}$ and $f(b) = 0$. Let E be a splitting field of $f(x)$ over F, and C be the integral closure of A in E. Then there are $\beta_i \in E$ such that $\beta_1 = b$ and $f(x) = \prod_{i=1}^n (x - \beta_i)$; in particular all β_i's are integral over $\mathfrak{a}$. Since $f(b) = 0$, we have that $m_{b,F}(x) | f(x)$, which implies that all the zeros of $m_{b,F}(x)$ are integral over $\mathfrak{a}$. Hence by the previous lemma all the coefficients of $m_{b,F}(x)$ are integral over $\mathfrak{a}$; in particular c_i' are integral over A and clearly they are in F. As A is integrally closed, we deduce that c_i's are in A. Altogether we have $c_i \in A$ and c_i is integral over $\mathfrak{a}$. So again by the previous lemma $c_i \in \sqrt{\mathfrak{a}}$; and claim follows. $\square$

Going-Down Theorem

Proposition 5. *Suppose B/A is an integral extension, B is an integral domain, and A is integrally closed. Suppose $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$ are in* $\operatorname{Spec}(A)$ *and $\mathfrak{q}_1 \in \operatorname{Spec}(B)$ such that $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Then there is $\mathfrak{q}_0 \in \operatorname{Spec}(B)$ such that $\mathfrak{q}_0 \subseteq \mathfrak{q}_1$ and $\mathfrak{q}_0 \cap A = \mathfrak{p}_0$.*

Proof. We have to focus on primes ideals in $\mathfrak{q}_1$; that means we need to localize at $\mathfrak{q}_1$. Let $S_{\mathfrak{q}_1} := B \setminus \mathfrak{q}_1$ and $S_{\mathfrak{p}_1} := A \setminus \mathfrak{p}_1$; notice that $S_{\mathfrak{p}_1} = S_{\mathfrak{q}_1} \cap A$. So we have $A_{\mathfrak{p}_1} \subseteq S_{\mathfrak{p}_1}^{-1} B \subseteq B_{\mathfrak{q}_1}$ (and $A_{\mathfrak{p}_1} \subseteq S_{\mathfrak{p}_1}^{-1} B$ is an integral extension).

Claim. Let $f : A_{\mathfrak{p}_1} \to B_{\mathfrak{q}_1}$. To prove the proposition, it is enough to show that $S_{\mathfrak{p}_1}^{-1} \mathfrak{p}_0$ is in the image of f^*.

Proof of Claim. If $f^*(\widetilde{\mathfrak{q}}_0) = S_{\mathfrak{p}_1}^{-1}\mathfrak{p}_0$, then there is $\mathfrak{q}_0 \in \mathrm{Spec}(B)$ such that $\widetilde{\mathfrak{q}}_0 = S_{\mathfrak{q}_1}^{-1}\mathfrak{q}_0$,

$$S_{\mathfrak{q}_1} \cap \mathfrak{q}_0 = \varnothing \qquad (\Rightarrow \mathfrak{q}_0 \subseteq \mathfrak{q}_1.)$$

$$S_{\mathfrak{q}_1}^{-1}\mathfrak{q}_0 \cap S_{\mathfrak{p}}^{-1}A = S_{\mathfrak{p}}^{-1}\mathfrak{p}_0 \qquad (x \in \mathfrak{p}_0 \Rightarrow \frac{x}{1} \in S_{\mathfrak{q}_1}^{-1}\mathfrak{q}_0 \Rightarrow x \in \mathfrak{q}_0,$$

$$x \in \mathfrak{q}_0 \cap A \Rightarrow \frac{x}{1} \in S_{\mathfrak{q}_1}^{-1}\mathfrak{q}_0 \cap S_{\mathfrak{p}}^{-1}A \Rightarrow \frac{x}{1} \in S_{\mathfrak{p}_1}^{-1}\mathfrak{p}_0 \Rightarrow x \in \mathfrak{p}_0.)$$

And so $\mathfrak{p}_0 = \mathfrak{q}_0 \cap A$ and $\mathfrak{q}_0 \subseteq \mathfrak{q}_1$.

We have proved earlier that a prime ideal is in the image of f^* if and only if it is a contracted ideal. This means it is enough to prove $(S_{\mathfrak{p}_1}^{-1}\mathfrak{p}_0)^{ec} = S_{\mathfrak{p}_1}^{-1}\mathfrak{p}_0$. Suppose to the contrary $\frac{a}{s} \in (S_{\mathfrak{p}_1}^{-1}\mathfrak{p}_0)^{ec} \setminus S_{\mathfrak{p}_1}^{-1}\mathfrak{p}_0$; so we are assuming there are $a \in A \setminus \mathfrak{p}_0$ and $s \in S_{\mathfrak{p}_1}$ such that $\frac{a}{s} \in \mathfrak{p}_0 B_{\mathfrak{q}_1} \setminus \mathfrak{p}_0 A_{\mathfrak{p}_1}$. This means there are $b_i \in B$, $s_i \in S_{\mathfrak{q}_1}$, and $p_i \in \mathfrak{p}_0$ such that

$$\frac{b}{s} = \sum_i p_i \frac{b_i}{s_i}.$$

This implies that for some $s_i', s' \in S_{\mathfrak{q}_1}$ we have

$$as' = \sum_i s_i' p_i b_i \in \mathfrak{p}_0^e.$$

Hence as' is integral over $\mathfrak{p}_0$. Therefore by the previous lemma,

$$m_{as',F}(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$$

for some $a_i \in \mathfrak{p}_0$ where F is the field of fractions of A. Notice that

$$m_{s',F}(x) = \frac{1}{a^n} m_{as',F}(ax),$$

as $a \in A \subseteq F$. We also notice that since s' is integral over A, by the previous lemma, $m_{s',F}(x) \in A[x]$; say $m_{s',F}(x) = x^m + a_{m-1}'x^{m-1} + \cdots + a_0'$. Then

$$a^i a_{m-i}' = a_{m-i} \in \mathfrak{p}_0.$$

As $a \notin \mathfrak{p}_0$ and $\mathfrak{p}_0$ is prime, we deduce that $a_{m-i}' \in \mathfrak{p}_0$. This means s' is integral over $\mathfrak{p}_0$; and so

$$s' \in \sqrt{\mathfrak{p}_0^e} \subseteq \sqrt{\mathfrak{p}_1^e} \subseteq \sqrt{\mathfrak{q}_1} = \mathfrak{q}_1,$$

which contradicts $s' \in S_{\mathfrak{q}_1}$. $\square$

Theorem 6 (Going-Down)**.** *Suppose B/A is an integral extension, B is an integral domain, and A is integrally closed. Suppose*

$$\begin{array}{ccccccccccc} & & & & \mathfrak{q}_m & \subsetneq & \cdots & \subsetneq & \mathfrak{q}_n & \in \operatorname{Spec}(B) \\ \mathfrak{p}_0 & \subsetneq & \cdots & \subsetneq & \mathfrak{p}_m & \subsetneq & \cdots & \subsetneq & \mathfrak{p}_n & \in \operatorname{Spec}(A), \end{array}$$

and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$. Then we can go down in the chain; that means there are $\mathfrak{q}_0 \subsetneq \cdots \mathfrak{q}_{m-1}$ in Spec(B) *such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.*

Proof. One can easily defined $\mathfrak{q}_i$'s inductively: for $m = n + 1$, one can use the surjectivity of f^*. And the induction step can be deduce by the above proposition. □

We will see how Going-Down can help us to show f^* is open.