INTEGRAL MORPHISMS ARE OPEN UNDER SOME CONDITIONS.

Suppose \( f : A \hookrightarrow B \) is an integral embedding; so far we have proved that \( f^* \) is onto and closed, any fiber has dimension 0, and \( \dim A = \dim B \). Next we proved the Going-Down Theorem under some conditions. Today we show that under the same conditions \( f^* \) is also open.

**Theorem 1.** Suppose \( f : A \hookrightarrow B \) is integral, \( B \) is an integral domain, and \( A \) is integrally closed. Then \( f^* : \text{Spec}(B) \to \text{Spec}(A) \) is open.

**Proof.** We know that \( \{\mathcal{O}_b\}_{b \in B} \) forms a basis for the open subsets of \( \text{Spec}(B) \) where \( \mathcal{O}_b := \{q \in \text{Spec}(B) \mid b \not\in q\} \). So it is enough to show \( f^*(\mathcal{O}_b) \) is open for any \( b \in B \). Take \( b \in B \), and let \( g(x) \) be the minimal polynomial of \( b \) over the field of fractions of \( A \); say \( g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \). In the previous lecture, we have proved that \( a_i \in A \).

**Claim.** \( f^*(\mathcal{O}_b) = \bigcup_{i=0}^{n-1} \mathcal{O}_{a_i} \); and so \( f^*(\mathcal{O}_b) \) is open.

**Proof of Claim.** (\( \subseteq \)) Suppose \( p \in f^*(\mathcal{O}_b) \). Then there is \( q \in \text{Spec}(B) \) such that \( q^c = p \); and so \( p^c \subseteq q \), which implies that \( \sqrt{p^c} \subseteq q \). Therefore knowing that \( b \not\in q \) implies that \( b \not\in \sqrt{p^c} \). By an earlier result, we get that \( b \) is not integral over \( p \); and so at least one of the \( a_i \)'s is not in \( p \), which means \( p \in \bigcup_{i=0}^{n-1} \mathcal{O}_{a_i} \).

(\( \supseteq \)) Suppose \( p \in \bigcup_{i=0}^{n-1} \mathcal{O}_{a_i} \); by a lemma that we proved in the previous lecture, if \( b \) is integral over \( p \), then all the non-leading coefficients of the minimal polynomial of \( b \) over the field of fractions of \( A \) should be in \( p \). So we deduce that \( b \) is not integral over \( p \). Thus by a proposition that was proved in the previous lecture, \( b \not\in \sqrt{p^c} \). Hence there is \( q \in \text{Spec}(B) \) such that \( p^c \subseteq q \) and \( b \not\in q \). So we have \( p \subseteq q^c \) is a chain in \( \text{Spec}(A) \); therefore by the Going-Down Theorem, there is \( q \in \text{Spec}(B) \) such that \( q \subseteq q^c \) and \( q^c = p \). Hence \( f^*(q) = p \) and \( b \not\in q \) as \( q \subseteq q^c \) and \( b \not\in q^c \); this means \( p \in f^*(\mathcal{O}_b) \). \( \square \)
Getting Noetherian condition for some integral closures.

As it has been pointed out earlier, one of the important examples that you should have in mind is the integral closure \( \mathcal{O}_k \) of \( \mathbb{Z} \) in a number field \( k \). So far we have proved that \( f^* : \text{Spec}(\mathcal{O}_k) \to \text{Spec}(\mathbb{Z}) \) is an open, closed, and onto map. And \( \dim \mathcal{O}_k = \dim \mathbb{Z} = 1 \). Next we want to show they are Noetherian. We do it in much more generality.

**Theorem 2.** Suppose \( f : A \hookrightarrow B \) is integral, \( B \) is an integral domain, and \( A \) is integrally closed. Let \( F \) be the field of fractions of \( A \), and \( E \) be the field of fractions of \( B \). Suppose \( E/F \) is a finite separable extension. Then there are \( e_1, \ldots, e_n \in E \) such that

\[
B \subseteq Ae_1 + \cdots + Ae_n.
\]

In particular, if \( A \) is Noetherian, then \( B \) is Noetherian.

Before we prove the claimed inclusion (1) in above theorem, we show how this implies the claimed Noetherian condition.

**Proof of the Noetherian condition.** If \( A \) is Noetherian, then any finitely generated \( A \)-module is a Noetherian \( A \)-module. Hence \( \sum_{i=1}^n Ae_i \) is a Noetherian \( A \)-module. This implies that any of its \( A \)-submodules is Noetherian; and so \( B \) is a Noetherian \( A \)-module. Therefore \( B \) is a Noetherian \( B \)-module, which means \( B \) is Noetherian. \(\Box\)

To show the above theorem, first we review some basic properties of finite separable field extensions and non-degenerate bilinear forms.

Recall from linear algebra. Suppose \( V \) is a finite dimensional vector space over a field \( F \). Let \( \mathfrak{B} := \{v_1, \ldots, v_n\} \) be an \( F \)-basis of \( V \). For \( v \in V \), we let

\[
|v\rangle_{\mathfrak{B}} := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}
\]

if \( v = \sum_{i=1}^n c_i v_i \); and we let \( \langle v\rangle_{\mathfrak{B}} \) be the transpose of \( |v\rangle_{\mathfrak{B}} \). For any \( F \)-linear map \( T : V \to V \), we have a matrix \( [T]_{\mathfrak{B}} \in M_n(F) \) such that for any \( v \in V \), \( |T(v)\rangle_{\mathfrak{B}} = [T]_{\mathfrak{B}}|v\rangle_{\mathfrak{B}} \).

**Lemma 3.** Suppose \( E/F \) is finite separable field extension. Let \( \mathfrak{B} := \{e_1, \ldots, e_n\} \) be an \( F \)-basis of \( E \), and \( \iota_e : E \to E, \iota_e(e') := ee' \). Let \( \{\sigma_1, \ldots, \sigma_n\} \) be the set of
$F$-embeddings of $E$ into an algebraic closure $\overline{F}$ of $F$. Then for any $e \in E$, $[l_e]_{\mathfrak{B}}$ is similar to the diagonal matrix $\text{diag}(\sigma_1(e), \ldots, \sigma_n(e))$ over $\overline{F}$.

**Proof.** Since $E/F$ is a finite separable extension, $E = F[\alpha]$ for some $\alpha \in E$. Let $m_{\alpha,F}(x)$ be the minimal polynomial of $\alpha$ over $F$. So $[E : F] = \deg m_{\alpha,F}$. Since $E/F$ is a separable extension, $m_{\alpha,F}(x)$ has $n := [E : F]$ distinct zeros; say $m_{\alpha,F}(x) = \prod_{i=1}^n (x - \alpha_i)$ for $\alpha_i$. Then for any $\sigma \in \text{Embed}_F(E, \overline{F})$, $\sigma(\alpha) \in \{\alpha_1, \ldots, \alpha_n\}$; and for any $i$, there is a unique $F$-embedding $\sigma_i$ of $E$ into $\overline{F}$ that sends $\alpha$ to $\alpha_i$. Hence after rearranging we can and will assume that $\sigma_i(\alpha) = \alpha_i$.

We have

$$E \otimes_F \overline{F} = F[\alpha] \otimes_F \overline{F}$$

$$\simeq F[x]/\left(\prod_{i=1}^n (x - \alpha_i)\right) \otimes_F \overline{F} \simeq \overline{F}[x]/\left(\prod_{i=1}^n (x - \alpha_i)\right)$$

$$\simeq \bigoplus_{i=1}^n \overline{F}[x]/\langle x - \alpha_i \rangle \simeq \bigoplus_{i=1}^n \overline{F}.$$}

And following these isomorphisms we have that

$$\alpha \otimes 1 \mapsto x + a \otimes 1 \mapsto x + a^e$$

$$\mapsto (x + \langle x - \alpha_1 \rangle, \ldots, x + \langle x - \alpha_n \rangle) \mapsto (\alpha_1, \ldots, \alpha_n)$$

$$= (\sigma_1(\alpha), \ldots, \sigma_n(\alpha)).$$

Since the above isomorphism is an $\overline{F}$-algebra isomorphism and $E = F[\alpha]$, we get an $\overline{F}$-algebra isomorphism

$$\theta : E \otimes_F \overline{F} \rightarrow \bigoplus_{i=1}^n \overline{F}, \theta(e \otimes 1) = (\sigma_1(e), \ldots, \sigma_n(e)),$$

for any $e \in E$. Therefore we get the following commuting diagram

$$\begin{array}{ccc}
E & \rightarrow & E \otimes_F \overline{F} \\
\downarrow l_e & & \downarrow l_e \otimes \text{id}_\overline{F} \\
E & \leftrightarrow & E \otimes_F \overline{F} \\
\end{array}
$$

where $d_e : \overline{F}^n \rightarrow \overline{F}^n, d_e(x_1, \ldots, x_n) := (\sigma_1(e)x_1, \ldots, \sigma_n(e)x_n)$. Notice that since $\{e_1, \ldots, e_n\}$ is an $F$-basis of $E$, $\{e_1 \otimes 1, \ldots, e_n \otimes 1\}$ is an $\overline{F}$-basis of $E \otimes_F \overline{F}$ and $\mathfrak{B} := \{\theta(e_1), \ldots, \theta(e_n)\}$ is an $\overline{F}$-basis of $\overline{F}^n$. As $\theta$ is an $\overline{F}$-algebra isomorphism,
by the above diagram $[l_e]_{\mathfrak{B}} = [d_e]_{\mathfrak{B}}$. On the other hand, in the standard basis $\mathfrak{B}'$ of $\overline{F}$ we have $[d_e]_{\mathfrak{B}'} = \diag(\sigma_1(e), \ldots, \sigma_n(e))$; and claim follows. \hfill \Box

**Corollary 4.** Suppose $E/F$ is a finite separable extension. Let

$$\text{Tr}_{E/F}(e) := \sum_{\sigma \in \text{Embed}_F(E,\overline{F})} \sigma(e),$$

and

$$\text{N}_{E/F}(e) := \prod_{\sigma \in \text{Embed}_F(E,\overline{F})} \sigma(e).$$

Let $l_e : E \rightarrow E, l_e(e') := ee'$. Then $\text{Tr}_{E/F}(e) = \text{Tr}(l_e)$ and $\text{N}_{E/F}(e) = \det l_e$; in particular, $\text{Tr}_{E/F}(E) \subseteq F$ and $\text{N}_{E/F}(E) \subseteq F$.

**Note.** Suppose $\mathfrak{B} := \{e_1, \ldots, e_n\}$ is an $F$-basis of a vector space $V$; and $f : V \times V \rightarrow F$ is a bilinear map. Then $[f]_{\mathfrak{B}} := [f(e_i, e_j)]$, and for any $v, w \in V$, we have $f(v, w) = \langle v|_{\mathfrak{B}} f \rangle_{\mathfrak{B}} [w]_{\mathfrak{B}}$.

**Lemma 5.** In the above setting, $f$ is non-degenerate if and only if $\det [f]_{\mathfrak{B}} \neq 0$.

**Proof.** ($\Rightarrow$) suppose $\det [f]_{\mathfrak{B}} = 0$; then there is $w \neq 0$ such that $[f]_{\mathfrak{B}} [w]_{\mathfrak{B}} = 0$; and so for any $v \in V$, $f(v, w) = 0$ and $w \neq 0$, which contradicts the assumption that $f$ is non-degenerate.

($\Leftarrow$) suppose $f$ is degenerate; so there is $w \neq 0$ such that $f(V, w) = 0$; this implies that $\langle v|_{\mathfrak{B}} f \rangle_{\mathfrak{B}} [w]_{\mathfrak{B}} = 0$ for any $v \in V$. Letting $v = e_i$, we deduce that the $i$-th component of $[f]_{\mathfrak{B}} [w]_{\mathfrak{B}}$ is zero. Therefore $[f]_{\mathfrak{B}} [w]_{\mathfrak{B}} = 0$. As $\det [f]_{\mathfrak{B}} \neq 0$, we deduce that $w = 0$, which is a contradiction. \hfill \Box

**Lemma 6.** Suppose $V$ is a finite dimensional $F$-vector space, and $f : V \times V \rightarrow F$ is a non-degenerate $F$-bilinear form; then $T_f : V \rightarrow V^*$, $(T_f(v))(w) := f(v, w)$ is an $F$-module isomorphism, where $V^* := \text{Hom}_F(V, F)$.

**Proof.** Since $f$ is linear in the second factor, $T_f(v) \in V^*$; and since $f$ is linear in the first factor, $v \mapsto T_f(v)$ is a linear map. If $v \in \ker T_f$, then for any $w \in V$, $(T_f(v))(w) = 0$, which implies that $f(v, V) = 0$; and so $v = 0$ as $f$ is non-degenerate. Hence $T_f$ is an injective $F$-linear map. On the other hand, $V^* = \text{Hom}_F(\bigoplus_{i=1}^n F, F) \cong \bigoplus_{i=1}^n \text{Hom}_F(F, F) \cong F^n \cong V$. Hence $T_f$ is also surjective as $V$ and $V^*$ have equal dimensions. \hfill \Box
Lemma 7. Suppose \( \{v_1, \ldots, v_n\} \) is an \( F \)-basis of \( V \), and \( f : V \times V \to F \) is a non-degenerate bilinear map. Then there is a dual basis \( \{w_1, \ldots, w_n\} \) with respect to \( f \); that means it is a basis and
\[
f(v_i, w_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}
\]
Proof. Let \( \{v_1^*, \ldots, v_n^*\} \) be the dual basis of \( V^* \); that means \( v_i^* : V \to F \) is the \( F \)-linear extension of \( v_i^*(v_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \) By the previous lemma, \( T_f \) is surjective; and so there are \( w_i \)'s in \( V \) such that \( T_f(w_i) = v_i^* \); and claim follows. \( \square \)

Lemma 8. Suppose \( E/F \) is a finite separable field extension. Then \( f(e, e') := \text{Tr}_{E/F}(ee') \) is a non-degenerate symmetric bilinear form on \( E \).

Proof. Suppose \( \{e_1, \ldots, e_n\} \) is an \( F \)-basis of \( E \). Then we have to show
\[
\det[\text{Tr}_{E/F}(e_i e_j)] \neq 0.
\]
We notice that \( \text{Tr}_{E/F}(e_i e_j) = \sum_{k=1}^n \sigma_k(e_i e_j) = \sum_{k=1}^n \sigma_k(e_i) \sigma(e_j) \) where \( \text{Embed}_F(E, \overline{F}) \).
Let \( X := [\sigma_k(e_i)] \) (the \( ik \)-th entry is \( \sigma_k(e_i) \)). Then by the previous equality we have
\[
[\text{Tr}_{E/F}(e_i e_j)] = XX^*; \text{ and so } \det[\text{Tr}_{E/F}(e_i e_j)] = \det X^2.
\]
Hence it is enough to show rows of \( X \) are linearly independent. This we have already pointed out in the proof of Lemma 3: \( \{\theta(e_1), \ldots, \theta(e_n)\} \) is an \( \overline{F} \)-basis of \( \overline{F}^n \). \( \square \)

We will prove Theorem 3 in the next lecture.