Lecture 1: definition.

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Let $\mathcal{H}$ be the upper-half plane, i.e. $\mathcal{H} := \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$.

We define a new metric on $\mathcal{H}$, and will study its geodesics and isometries.

Hyperbolic metric. $ds = \dfrac{\sqrt{dx^2 + dy^2}}{y}$, i.e. for any C^1- curve

$\gamma : [0,1] \longrightarrow \mathcal{H}$, $\gamma(t) = x(t) + i\, y(t)$,

$$\ell_{\mathcal{H}}(\gamma) := \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt.$$

Remark. It is a Riemannian metric: at the point $z = x + iy$ after identifying the tangent plane $T_z\mathcal{H}$ with the underlying plane, the dot product is $\frac{1}{y} I$.

Hyperbolic distance. $$d_{\mathcal{H}}(z_1, z_2) := \inf_{\substack{\gamma(0) = z_1 \\ \gamma(1) = z_2 \\ \gamma : C^1}} \ell_{\mathcal{H}}(\gamma).$$

Ex. Let $\gamma : [a,b] \longrightarrow \mathcal{H}$, $\gamma(t) = it$ where $a, b \in \mathbb{R}^+$.

What is $\ell_{\mathcal{H}}(\gamma)$?

Solution. $\ell_{\mathcal{H}}(\gamma) := \int_a^b \frac{1}{t}\, dt = \ln(b/a)$. ■

Qu. What is $d_{\mathcal{H}}(ia, ib)$?

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Solution. Let $\gamma : [0,1] \to \mathcal{H}$, $\gamma(0) = ia$, $\gamma(1) = ib$.

$$\Rightarrow \ell_{\mathcal{H}}(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt \overset{\circledast}{\geq} \int_0^1 \frac{|y'(t)|}{y(t)}\, dt$$

$$\geq \ln y(t) \Big|_0^1 = \ln(b/a).$$

$$\Rightarrow d_{\mathcal{H}}(ia, ib) = \ln(b/a). \quad \blacksquare$$

Remark. At $\circledast$ equality holds if and only if $x'(t) = 0$ (a.e.)

$\rightarrow \gamma$ is a segment.

Geodesic $\operatorname{Im}(\gamma)$ of $\gamma : \mathbb{R} \to \mathcal{H}$ is called a geodesic if

$$d_{\mathcal{H}}(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|.$$

Cor. Half-line connecting 0 to "∞", ie. perpendicular to the real axis is a geodesic. It is the unique geodesic that passes through two of its points.

Isometries

$\mathcal{H}$ has a rich group of isometries. Recall that

$$SL_2(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \;\middle|\; ad - bc = 1 \right\}$$

acts on $\mathcal{H}$ via Möbius transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d}.$$

<u>Why?</u> $$\omega = \frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}$$

$$= \frac{ac\,|z|^2 + ad\,z + bc\,\bar{z} + bd}{|cz+d|^2}.$$

$$\Rightarrow \operatorname{Im}(\omega) = \frac{\omega - \bar{\omega}}{2i} = \frac{(ad-bc)(z-\bar{z})}{(2i)\,|cz+d|^2}.$$

$$= \operatorname{Im}(z)/|cz+d|^2 .$$

$$\boxed{\operatorname{Im}(\omega) = \frac{\operatorname{Im}(z)}{|cz+d|^2}}$$