Lecture 1: definition.

Let $\mathcal{H}$ be the upper-half plane, i.e. $\mathcal{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$. We define a new metric on $\mathcal{H}$, and will study its geodesics and isometries.

**Hyperbolic metric.** $\ ds = \frac{\sqrt{dx^2 + dy^2}}{y}$, i.e. for any $C^1$-curve $\gamma: [0,1] \rightarrow \mathcal{H}$, $\gamma(t) = x(t) + iy(t)$,

$$ \ell_{\mathcal{H}}(\gamma) := \int_0^1 \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)} \ dt. $$

**Remark.** It is a Riemannian metric: at the point $z = x + iy$ after identifying the tangent plane $T_z \mathcal{H}$ with the underlying plane, the dot product is $\frac{1}{y} \ I$.

**Hyperbolic distance.** $\ d_{\mathcal{H}}(z_1, z_2) := \inf_{\gamma} \ell_{\mathcal{H}}(\gamma)$.

**Ex.** Let $\gamma: [a,b] \rightarrow \mathcal{H}$, $\gamma(t) = it$ where $a,b \in \mathbb{R}^+$. What is $\ell_{\mathcal{H}}(\gamma)$?

**Solution.** $\ell_{\mathcal{H}}(\gamma) := \int_a^b \frac{1}{t} \ dt = \ln \left( \frac{b}{a} \right)$. □

**Qu.** What is $d_{\mathcal{H}}(ia, ib)$?
Qu. What is $d_H(i\alpha, i\beta)$?

Solution. Let $\gamma : [0, 1] \to H$, $\gamma(0) = i\alpha$, $\gamma(1) = i\beta$.

$$
\Rightarrow \ell_H(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} \, dt \geq \int_0^1 \frac{1}{y(t)} \, dt \\
\geq \ln y(t) \bigg|_0^1 = \ln \left(\frac{b}{a}\right).
$$

$$
\Rightarrow d_H(i\alpha, i\beta) = \ln \left(\frac{b}{a}\right). \quad \square
$$

Remark. At $\bigcirc$ equality holds if and only if $x(t) = 0 \ (a.e.)$

$\Rightarrow \gamma$ is a segment.

Geodesic $\text{Im}(\gamma)$ of $\gamma : \mathbb{R} \to H$ is called a geodesic if

$$
d_H(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|.
$$

Cor. Half-line connecting $0$ to $\text{"oo"}$, i.e. perpendicular to the real axis is a geodesic. It is the unique geodesic that passes through two of its points.

Isometries

$H$ has a rich group of isometries. Recall that

$$
\text{SL}_2(\mathbb{R}) := \{ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in M_2(\mathbb{R}) \mid ad - bc = 1 \}.
$$

acts on $H$ via Möbius transformations.
\[
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} . z := \frac{az + b}{cz + d}.
\]

**Why?**

\[
\omega = \frac{az + b}{cz + d} = \frac{(az + b)(cz + d)}{|cz + d|^2} = \frac{ac |z|^2 + ad \, z + bc \bar{z} + bd}{|cz + d|^2}.
\]

\[
\Rightarrow \quad \text{Im}(\omega) = \frac{\omega - \overline{\omega}}{2i} = \frac{(ad - bc)(z - \bar{z})}{(2i) |cz + d|^2} = \frac{\text{Im}(z)}{|cz + d|^2}.
\]

\[
\text{Im}(\omega) = \frac{\text{Im}(z)}{|cz + d|^2}.
\]