Lecture 2: geodesics and isometries

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<u>Lemma</u>. $PSL_2(\mathbb{R}) \hookrightarrow \mathrm{Isom}(\mathcal{H})$ via the Möbius transformations.

<u>Pf</u>. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$, and $T_g : \mathcal{H} \to \mathcal{H}$, $T_g(z) = \dfrac{az+b}{cz+d}$.

We need to show that $\ell_{\mathcal{H}}(\gamma) = \ell_{\mathcal{H}}(T_g \circ \gamma)$ for any C^1-curve $\gamma : [0,1] \to \mathcal{H}$.

$$\ell_{\mathcal{H}}(T_g \circ \gamma) = \int_0^1 \frac{\left|\frac{d}{dt}\Big|_t (T_g \circ \gamma)\right|}{|\mathrm{Im}\,(T_g(\gamma(t)))|}\, dt$$

- By chain rule for complex-valued functions we have

$$\frac{d}{dt}(T_g \circ \gamma)\Big|_t = T_g'(\gamma(t))\, \gamma'(t) \qquad \text{(complex multiplication)}$$

$$= \frac{\gamma'(t)}{(c\gamma(t)+d)^2}.$$

- We have seen that

$$\mathrm{Im}(T_g(\gamma(t))) = \frac{\mathrm{Im}(\gamma(t))}{|c\gamma(t)+d|^2}.$$

So we have

$$\ell_{\mathcal{H}}(T_g \circ \gamma) = \int_0^1 \frac{|\gamma'(t)| / |c\gamma(t)+d|^2}{\mathrm{Im}(\gamma(t)) / |c\gamma(t)+d|^2}\, dt$$

$$= \int_0^1 \frac{|\gamma'(t)|}{\mathrm{Im}\,\gamma(t)}\, dt = \ell_{\mathcal{H}}(\gamma). \quad \blacksquare$$

Lemma. For any $\xi_0 < \xi_1 < \xi_2 \in \partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$, there is a unique $[g] \in PSL_2(\mathbb{R})$ s.t. $g \cdot 0 = \xi_0$, $g \cdot 1 = \xi_1$, and $g \cdot \xi_2 = \infty$.

Pf. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose $\xi_2 \neq \infty$.

$$\left.\begin{aligned} b/d &= \xi_0 \\ \frac{a+b}{c+d} &= \xi_1 \\ a/c &= \xi_2 \end{aligned}\right\} \Rightarrow \begin{aligned} &[\text{Since } \xi_0 \neq \infty, \text{ we have } d \neq 0.] \\ &a/d + \xi_0 = \xi_1 (c/d + 1) \\ \Rightarrow\ &\xi_2\, c/d + \xi_0 = \xi_1\, c/d + \xi_1 \\ \Rightarrow\ &c/d = (\xi_1 - \xi_0)/(\xi_2 - \xi_1) =: \eta > 0 \end{aligned}$$

$$\Rightarrow g = d \begin{bmatrix} \xi_2 \eta & \xi_0 \\ \eta & 1 \end{bmatrix} \quad \text{and} \quad \det(g) = d^2 \eta (\xi_2 - \xi_0) > 0$$

$\Rightarrow$ $g/\sqrt{\det(g)} \in SL_2(\mathbb{R})$ does the job, and $[g/\sqrt{\det(g)}] \in PSL_2(\mathbb{R})$ is the unique such element. $\blacksquare$

Cor. Any semi-line and semi-circle perpendicular to the real axis is a hyperbolic geodesic.

Pf. Let ℓ be such a semi-line or semi-circle. Let $\ell_{-\infty} < \ell_{+\infty}$ be the end-points of ℓ. Then there is $[g] \in PSL_2(\mathbb{R})$ s.t. $g \cdot 0 = \ell_{-\infty}$ and $g \cdot \infty = \ell_{+\infty}$.

Any Möbius transformation sends a line or a circle to a

Any Möbius transformation sends a line or a circle to a line or a circle and <u>preserves</u> the angle between them (why?). So g sends the semi-line $i[0,\infty]$ to ℓ.

Since $i[0,\infty]$ is a hyperbolic geodesic and $[g] \in \mathrm{Isom}(\mathcal{H})$, we have that ℓ is a hyperbolic geodesic. ■

<u>Cor</u>. Any hyperbolic geodesic is of the above form.

<u>Pf</u>. Let ℓ be a hyperbolic geodesic. Let ℓ' be an either semi-line or semi-circle that passes through two points A & B of ℓ. By the above corollary, there is $[g] \in PSL_2(\mathbb{R})$ s.t.

$[g] \cdot \ell' =$ the geodesic connecting 0 to ∞.

So $[g] \cdot \ell$ is a hyperbolic geodesic which passes through $[g] \cdot A$ and $[g] \cdot B$. Since $[g] \cdot A = ia$ and $[g] \cdot B = ib$, there is a unique hyperbolic geodesic connecting them.

So $[g] \cdot \ell = [g] \cdot \ell'$, which implies $\ell = \ell'$. ■

Notice that $\begin{bmatrix} e^{t/2} & \\ & e^{-t/2} \end{bmatrix} \cdot z = e^t z$ is a hyperbolic isom.

and it sends the hyperbolic geodesic $[0,\infty]$ to itself and translates on this line. In particular for any $a \in \mathbb{R}^+$ we have that $(0, i, \infty) \longmapsto (0, ia, \infty)$

$$z \longmapsto \begin{bmatrix} \sqrt{a} & \\ & \sqrt{a}^{-1} \end{bmatrix} \cdot z.$$

<u>Cor.</u> For any hyperbolic geodesic ℓ and $p \in \ell$, there is $[g] \in PSL_2(\mathbb{R})$ s.t. $[g] \cdot i\mathbb{R}^+ = \ell$ and $[g] \cdot i = p$.