Lecture5: hyperbolic geometry and symmetric space.

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Lemma $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \xrightarrow{\sim} \mathcal{H}$

Pf. Consider the Möbius transformation map:

It is a transitive action.

$\mathrm{Stab}_{SL_2(\mathbb{R})}(i) := \{g \in SL_2(\mathbb{R}) \mid g.i = i\}$ $\quad -c \ \ d]$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot i = i \Rightarrow \frac{a\, i + b}{c\, i + d} = i$$

$$\Rightarrow a\, i + b = -c + d\, i$$

$$\Rightarrow a = d \text{ and } b = -c$$

$$\Rightarrow g = \begin{bmatrix} a & b \\ -b & \end{bmatrix} \in SO_2(\mathbb{R}).$$

So $g\, SO_2(\mathbb{R}) \longmapsto g.i$ is a bijection.

$g_n SO_2(\mathbb{R}) \xrightarrow[n\to\infty]{} g\, SO_2(\mathbb{R}) \Rightarrow \exists\, k_n \in SO_2(\mathbb{R}),$ $\quad n\to\infty$

$$\Rightarrow g_n k_n \cdot i = g_n \cdot i \longrightarrow g \cdot i.$$

$g_n \cdot i \xrightarrow[n\to\infty]{} g \cdot i \Rightarrow (g^{-1} g_n) \cdot i$ $\quad n\to\infty$

$$\Rightarrow \frac{a_n d_n - b_n c_n}{c_n^2 + d_n^2}\, i + \frac{a_n c_n + b_n d_n}{c_n^2 + d_n^2} \quad n\to\infty$$

$$\Rightarrow c_n^2 + d_n^2 \xrightarrow[n\to\infty]{} 1$$

$$a_n c_n + b_n d_n \xrightarrow[n\to\infty]{} 0$$

$n\to\infty$

Let $k_n = \frac{1}{\sqrt{c_n^2+d_n^2}}\begin{bmatrix} d_n & c_n \\ -c_n & d_n\end{bmatrix} \in SO_2(\mathbb{R})$. Then

$$g^{-1}g_n k_n = \begin{bmatrix} \sqrt{c_n^2+d_n^2} & (a_nc_n+b_nd_n)/\sqrt{c_n^2+d_n^2} \\ 0 & \sqrt{c_n^2+d_n^2}^{\,-1}\end{bmatrix} \xrightarrow[n\to\infty]{} I$$

$$\Rightarrow g^{-1}g_n\, SO_2(\mathbb{R}) \longrightarrow SO_2(\mathbb{R}). \quad \blacksquare$$

[An alternative approach which is less computational:

$$d_{\mathcal{H}}(g_n\cdot i, i) = d_{\mathcal{H}}(k_n^{(1)} a_n k_n^{(2)} \cdot$$

$$= d_{\mathcal{H}}(a_n\cdot i, i) \xrightarrow[n\to\infty]{}$$

where $k_n^{(1)}, k_n^{(2)} \in SO_2(\mathbb{R})$ and $\begin{bmatrix} \lambda_n & \\ & \lambda_n^{-1}\end{bmatrix}$.

$$d_{\mathcal{H}}(a_n\cdot i, i) = 2\,|\log \lambda_n| \to 0 \Rightarrow$$

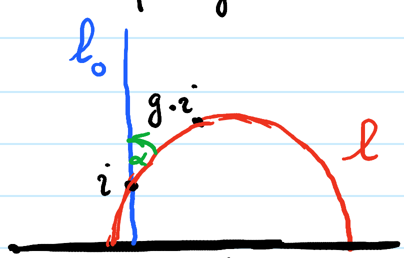
$$\Rightarrow k_n^{(1)} a_n k_n^{(1)^{-1}} \longrightarrow I$$

$$\Rightarrow g_n \cdot k_n^{(2)^{-1}} \cdot k_n^{(1)^{-1}} \longrightarrow$$

$$\Rightarrow g\ SO_2(\mathbb{R}) \longrightarrow SO_2(\mathbb{R}).\ ]$$

[A geometric way of thinking about KAK decomposition for $SL_2(\mathbb{R})$:

For any $g \in G\setminus K$, $g\cdot i \neq i$. So there is a unique geodesic from i to $g\cdot i$.



So if we rotate ℓ by angle $\underline{\alpha}$ about i, we get the geodesic ℓ_0. Hence, for some $k \in K$, $k\cdot(g\cdot i)$ is on

ℓ_0, which implies $\quad k\cdot(g\cdot i) = a\cdot i$ for some diag. matrix a. So $(a^{-1}kg)\cdot i = i \Rightarrow a^{-1}kg = k' \in K$

$\Rightarrow g = k^{-1} a k'$.]

Another important way to view $\qquad {}_{SO_2(\mathbb{R})}$ is identifying L with a subset of positive definite

<u>Proposition</u> Let $P := \{ X \in SL_n(\mathbb{R}) \mid X \text{ is positive definite} \}$, $K = SO_n(\mathbb{R})$, and $G = SL_n(\mathbb{R})$. Then

$$\eta : G/K \longrightarrow P, \quad \eta(gK) := gg^t$$

is a homeomorphism.

<u>Pf of prop.</u> ① <u>well-defined</u>. • $(gk)(gk)^t = g\,kk^t g^t = gg^t$ for any $k \in K$.

• $\forall\, x \in \mathbb{R}^n \setminus \vec{0}$, $\quad x^t g g^t x = \|g^t x\|^2 > 0$. So gg^t is positive definite.

② <u>1-1</u>. $\eta(g_1K) = \eta(g_2K) \Rightarrow g_1 g_1^t = g_2 g_2^t \Rightarrow (g_2^{-1} g_1)(g_2^{-1} g_1)^t = I$

$\Rightarrow g_2^{-1} g_1 \in K \Rightarrow g_1 K = g_2 K$.

③ <u>Onto</u>. $p \in P \Leftrightarrow \exists\, k \in K$, a : diag. with positive entries s.t. $p = k a k^t$

$\left(\Leftrightarrow p = \exp(X) \text{ for some } \underbrace{X \in \mathfrak{sl}_n(\mathbb{R})}_{tr(X)=0},\ X = X^t.\right)$

$\Big(\quad \cdot \quad \cdot \quad = \exp(X) \quad \text{for some } \underbrace{X \in \mathfrak{sl}_n(\mathbb{R})}_{tr(X)=0}, \; X = X^t \Big)$

So $p = (k\sqrt{a}\,k^t)(k\sqrt{a}\,k^t)^t = \eta(k\sqrt{a}K)$.

④ continuous.

$$g_nK \longrightarrow K \Rightarrow \exists k_n \in K,\ g_nk_n \longrightarrow I \Rightarrow \eta(g_nK) = (g_nk_n)(g_nk_n)^t \longrightarrow \eta(K)$$

⑤ open.

$$g_n g_n^t \longrightarrow I \Rightarrow k_n^{(1)} a_n a_n k_n^{(1)\,t} \longrightarrow I$$

where $g_n = k_n^{(1)} a_n k_n^{(2)}$, $k_n^{(i)} \in SO_2(\mathbb{R})$ and a_n is diagonal with positive entries.

$\Rightarrow a_n \longrightarrow I$ (Since their eigenvalue goes to 1 and they are diag.)

$$\Rightarrow k_n^{(1)} a_n k_n^{(1)^{-1}} = g_n k_n^{(2)^{-1}} k_n^{(1)^{-1}} \longrightarrow I. \qquad \blacksquare$$

- $G \curvearrowright P$, $g \cdot p := gpg^t$ and $\eta : G/K \longrightarrow P$ is G-equivariant, i.e. $g \cdot \eta(g'K) = \eta(gg'K)$.
- We can identify G/K with P. And the tangent space at any point $\eta(g) \in P$ can be identified with

$$\mathfrak{p} := \{ X \in \mathfrak{sl}_n(\mathbb{R}) \mid X = X^t \}$$

as $\log : P \longrightarrow \mathfrak{p}$ and $\exp : \mathfrak{p} \longrightarrow P$ are well-defined (analytic) maps.

- Since K fixes $\eta(K)$, it acts on the tangent space:

$$\forall k \in K,\ X \in \mathfrak{p},\ k \cdot X \longmapsto kXk^t$$

$\forall\, k \in K,\ X \in \mathfrak{p},\quad k\cdot X \longmapsto kXk$

For $X, Y \in \mathfrak{p}$, let $\langle X, Y\rangle := \operatorname{tr}(XY) \underset{Y=Y^t}{=} \sum_{i,j} x_{ij}\, y_{ij}$.

Then $\langle\, ,\rangle$ is a dot-product on $\mathfrak{p}$ and $\langle k\cdot X, k\cdot Y\rangle = \langle X, Y\rangle$.

- For any $g_0K \in G/K$, $g_0Kg_0^{-1}$ fixes $\eta(g_0K)$ and so we get an action on $T_{\eta(g_0K)}(\mathcal{P})$ as before:

  $$g_0kg_0^{-1}\cdot X := (g_0kg_0^{-1})(X)(g_0kg_0^{-1})^t.$$

  This time the dot-product on $\mathfrak{p}$ should be $g_0Kg_0^{-1}$-invariant:

  $$\langle (g_0kg_0^{-1})\cdot X, (g_0kg_0^{-1})\cdot Y\rangle_{\eta(g_0K)} = \langle X, Y\rangle_{\eta(g_0K)}$$

  Let $\langle X, Y\rangle_{\eta(g_0K)} := \langle g_0^{-1}\cdot X, g_0^{-1}\cdot Y\rangle$.

  (Since $\langle \cdot, \cdot\rangle$ is K-invariant, it is well-defined.)

  So
  $$\begin{aligned}\langle (g_0kg_0^{-1})\cdot X, (g_0kg_0^{-1})\cdot Y\rangle_{\eta(g_0K)} &= \langle (kg_0^{-1})\cdot X, (kg_0^{-1})\cdot Y\rangle \\ &= \langle g_0^{-1}\cdot X, g_0^{-1}\cdot Y\rangle \\ &= \langle X, Y\rangle.\end{aligned}$$

  So by definition
  $$\begin{aligned}\langle X, Y\rangle_{p} &:= \operatorname{tr}(\sqrt{p}^{-1} X \sqrt{p}^{-1} \sqrt{p}^{-1} Y \sqrt{p}^{-1}) \\ &= \operatorname{tr}(p^{-1} X p^{-1} Y)\end{aligned}$$

  defines a Riemannian geometry on $\mathcal{P}$ which is G-invariant.

  ($p = \eta(\sqrt{p}K)$ and $\sqrt{p} = \sqrt{p}^t$ are used.)

($q = \eta(\sqrt{P}K)$ and $\sqrt{P} = \sqrt{P}^t$ are used.)

Hence length of a C^1-curve $\gamma : [0,1] \to \mathcal{P}$ is

$$\ell(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \, dt = \int_0^1 \sqrt{tr(\gamma(t)^{-1}\gamma'(t))^2} \, dt .$$

- Let $\Theta : G \to G$, $\Theta(g) = (g^t)^{-1}$. (Cartan involution)
  - $k \in K \iff \Theta(k) = k$.
  - Θ is an involution.
  - $d\Theta(X) = -X^t$ is an involution of $\mathfrak{g}$ and so

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+ \quad \text{where}$$

$$\mathfrak{g}_- = \{X \in \mathfrak{g} \mid d\Theta(X) = -X\} = \mathfrak{p}$$

and $\mathfrak{g}_+ = \{X \in \mathfrak{g} \mid d\Theta(X) = X\} = \mathfrak{k} = Lie(K)$.

- $(\Theta \circ \gamma)(t) = \gamma(t)^{-1} \Rightarrow \frac{d}{dt}(\Theta \circ \gamma)\Big|_{t=t_0} = -\gamma(t_0)^{-1}\gamma'(t_0)\gamma(t_0)^{-1}$

$$\Rightarrow tr\left((\Theta\circ\gamma)(t)^{-1} \frac{d}{dt}(\Theta\circ\gamma)(t)\right)^2 = tr\left(\cancel{\gamma(t)}\left[-\cancel{\gamma(t)^{-1}}\gamma'(t)\gamma(t)^{-1}\right]\right)^2$$

$$= tr(-\gamma'(t)\gamma(t)^{-1})^2$$

$$= tr(\gamma(t)^{-1}\gamma'(t))^2$$

$\Rightarrow \Theta \in Isom(\mathcal{P})$. By the transitivity of $G \subseteq Isom(\mathcal{P})$:

For any $p \in \mathcal{P}$, $\exists\, \Theta_p : \mathcal{P} \to \mathcal{P}$ s.t. $\Theta_p(p) = p$,

$\Theta_p^2 = id.$,

$$d\theta_p(X) = -X \quad \text{for } X \in \mathfrak{p},$$

$$\theta_p \in \mathrm{Isom}(\mathcal{P})$$

Such Riemannian space is called a symmetric Riemannian space.