In the previous lecture we learned 3 rules for limits of multivariable functions:

**Rule 1.** Function is nice; \( \lim_{t \to t_0} g(t) = L \) it is the answer. You plug in and no problem arises.

**Rule 2.**
\[
\lim_{(x,y) \to (a,b)} f(x,y) = t_0 \implies \lim_{(x,y) \to (a,b)} g(f(x,y)) = L
\]
\[
\lim_{t \to t_0} g(t) = L
\]

**Rule 3.** Approach \((a,b)\) via various lines and check if
\[
\lim_{x \to a} f(x, k(x-a)+b)
\]
depends on \(k\) or NOT.

If it depends on \(k\), then \(\lim_{(x,y) \to (a,b)} f(x,y)\)
does NOT exist.

**Warning.** If in Rule 3 the considered limit is independent of \(k\), you cannot conclude anything.

**Rule 4.** Try other curves to approach \((a,b)\). A good place to start for approaching \((0,0)\) is considering curves of the form \(x = y^c\) or \(y = x^c\).
Ex. Determine if the following limit exist.

\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3 y}{x^6 + y^2}.
\]

solution. [Clearly we cannot use rule 1 and rule 2]

Approaching \((0,0)\) along the line \(y = kx:\)

\[
\lim_{{x \to 0}} \frac{x^3 (kx)}{x^6 + k^2 x^2} = \lim_{{x \to 0}} \frac{k x^2}{x^4 + k^2} = 0,
\]

Since it is independent of \(k\), we cannot conclude anything.

Let's approach \((0,0)\) along a curve of the form \(y = x^c\).

We choose \(c\) such that numerator and denominator have the same degree: \(c = 3\).

\[
\lim_{{x \to 0}} \frac{x^3 \cdot x^3}{x^6 + (x^3)^2} = \frac{1}{2}.
\]

Since \(\frac{1}{2} \neq 0\), \(\lim_{{(x,y) \to (0,0)}} \frac{x^3 y}{x^6 + y^2}\) does not exist.

Ex. Suppose the contour diagram of \(f(x,y)\) looks like

Determine whether \(\lim_{{(x,y) \to (0,0)}} f(x,y)\) exist.
Solution Limit does NOT exist since we can approach (0,0) along different level curves. Along the curve \( f(x,y) = 1 \), the limit is 1, and along the curve \( f(x,y) = -1 \) the limit is -1. So \( \lim_{(x,y) \to (0,0)} f(x,y) \) does NOT exist.

If two level curves \( f(x,y) = c_1 \) and \( f(x,y) = c_2 \) can approach to the point \((a,b)\), then

\[
\lim_{(x,y) \to (a,b)} f(x,y) \text{ does NOT exist.}
\]

Rule 3 and 4 are useful to show a limit does NOT exist.

Rule 5. Use Squeeze Theorem:

If \( g_1(x,y) \leq f(x,y) \leq g_2(x,y) \) and

\[
\lim_{(x,y) \to (a,b)} g_1(x,y) = \lim_{(x,y) \to (a,b)} g_2(x,y) = L,
\]

then

\[
\lim_{(x,y) \to (a,b)} f(x,y) = L.
\]

Ex. Determine if the following exists. If it does, find its value.

\[
\lim_{(x,y) \to (0,0)} \sin(xy) \cos\left(\frac{1}{x^2+y^2}\right).
\]
Solution. When we try to plug in, we realize that

\[
\lim_{(x,y) \to (0,0)} \sin(xy) = 0, \quad \text{but} \quad \lim_{(x,y) \to (0,0)} \cos \left( \frac{1}{x^2 + y^2} \right) \\
\text{does NOT exist.}
\]

We have

\[
| \sin(xy) \cos \left( \frac{1}{x^2 + y^2} \right) | \leq | \sin(xy) |
\]

as \( | \cos \left( \frac{1}{x^2 + y^2} \right) | \leq 1 \). Hence

\[
-| \sin(xy) | \leq \sin(xy) \cos \left( \frac{1}{x^2 + y^2} \right) \leq | \sin(xy) |
\]

and

\[
\lim_{(x,y) \to (0,0)} \pm | \sin(xy) | = 0. \quad \text{Therefore by the squeeze theorem}
\]

\[
\lim_{(x,y) \to (0,0)} \sin(xy) \cos \left( \frac{1}{x^2 + y^2} \right) = 0.
\]

By a similar argument we have:

If \( \lim_{(x,y) \to (a,b)} f(x,y) = 0 \) and \( g(x,y) \) is bounded,

then \( \lim_{(x,y) \to (a,b)} f(x,y) g(x,y) = 0 \).

Important method: Using polar coordinates.

First notice \( \lim_{(x,y) \to (a,b)} f(x,y) = \lim_{(x,y) \to (0,0)} f(x+a, y+b) \).

So one has to understand limits where \( (x,y) \) approaches \( (0,0) \).
Writing $x$ and $y$ in polar coordinates means

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

where $r$ is the distance of $(x,y)$ from the origin, i.e. $r = \sqrt{x^2 + y^2}$ and $\theta$ is the angle that the segment $OP$ makes with the $x$-axis. So, as $(x,y) \to (0,0)$, $r \to 0$.

Hence \[
\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{r \to 0} f(r \cos \theta, r \sin \theta) \quad \text{uniform on } \theta.
\]

The best way to understand the phrase "uniform on $\theta$" is thinking about $\theta$ as an unknown function of $r$.

So we end up with reducing the two-variable limit to a single-variable limit with the caveat that we do not know what $\Theta(r)$ is:

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{r \to 0} f(r \cos \Theta(r), r \sin \Theta(r)).$$

We will see a general example of this type next time.