Lecture 14: Differentiation

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In the previous lecture we defined a differentiable function:

\[ f \text{ is differentiable at } (x_0,y_0) \text{ if } \lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - L(x,y)}{(x-x_0)^2 + (y-y_0)^2} = 0 \]

where \( L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) \).

As we have seen before, it is not very easy to deal with limits of multi-variable functions. Lucky us the following theorem can help us to get differentiability without dealing with limits!

**Theorem**  If in a disk around a point \( x_0 \) all the partial derivatives are continuous, then \( f \) is differentiable at \( x_0 \).

**Ex.** Find all the points where \( f(x,y) = \sqrt{x^2+y^2} \) is differentiable.

**Solution.** (As we have seen before) \( \frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \) and \( \frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}} \). So \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are continuous everywhere except possibly at \( (0,0) \). Hence for any \( (x_0,y_0) \neq (0,0) \), \( f \) has continuous partial derivative in a disk centered at \( (x_0,y_0) \) with radius smaller than \( \sqrt{x_0^2+y_0^2} \), e.g. \( \frac{1}{2} \sqrt{x_0^2+y_0^2} \). Therefore the above theorem implies that \( f \) is differentiable at \( (x_0,y_0) \). Next we notice
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\[ f(x, 0) = \sqrt{x^2} = |x| \text{ is not differentiable at } x=0 \text{ as a single-variable function. So } f_x \text{ does NOT exist at } (0,0), \text{ which implies } f \text{ is not differentiable at } (0,0). \]

\[ z = \sqrt{x^2 + y^2} \text{ is equation of a cone which clearly does NOT have a tangent plane at } (0,0,0). \]

**Ex.** Find equation of the tangent plane at \((1,1,\sqrt{2})\) in the above example.

**Solution.** 

\[ z = \sqrt{2} + f_x(1,1) (x-1) + f_y(1,1) (y-1) \]

\[ = \sqrt{2} + \frac{1}{\sqrt{2}} (x-1) + \frac{1}{\sqrt{2}} (y-1) \]

\[ z = \frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y . \]

**Ex.** Suppose \( f(x,y) = \frac{1}{\pi} \cos \left( \frac{\pi}{2} x^2 y \right) \). Find linear approximation of \( f(-1,1,1.2) \).

**Solution.** 

\[ f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0) (x-x_0) + f_y(x_0,y_0) (y-y_0) \]

\[ f_x(x,y) = -xy \sin \left( \frac{\pi}{2} x^2 y \right) \quad \text{and} \quad f_y(x,y) = -\frac{x^2}{2} \sin \left( \frac{\pi}{2} x^2 y \right) \]

\[ f(-1,1) = \frac{1}{\pi} \cos \left( \frac{\pi}{2} \right) = 0, \]

\[ f_x(-1,1) = \sin \left( \frac{\pi}{2} \right) = 1 \quad \text{and} \quad f_y(-1,1) = \frac{1}{2} \sin \left( \frac{\pi}{2} \right) = \frac{1}{2} \]
So \( f(x,y) \approx (x+1) - \frac{1}{2} (y-1) \), which implies

\[
f(-1.1, 1.2) \approx (-0.1) - \frac{1}{2} (0.2) = -0.2.
\]