In the previous lecture we studied single-variable vector-valued functions. We saw:

\[ \lim_{t \to t_0} \mathbf{r}(t) = \left( \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right) \]

\[ \mathbf{r}'(t) = (x'(t), y'(t), z'(t)) \]

Parametrization of tangent line of \( \mathbf{r}(t) \) at \( \mathbf{r}'(t_0) \) is given by:

\[ l(t) = \mathbf{r}(t_0) + t \mathbf{r}'(t_0) \]

Ex. Parametrize the circle of radius 2, centered at the origin, in the yz-plane.

Solution. Since it is in the yz-plane, \( x = 0 \).

\[ y = \cos \theta, \quad z = \sin \theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi. \]

So we get:

\[ \mathbf{r}^{\prime}(\theta) = (0, \cos \theta, \sin \theta). \]

Ex. Parametrize the curve of intersection of:

\[ x^2 + y^2 = 4 \quad \text{and} \quad x + y + z = 1 \]

Solution. Whenever you see an equation of the form, \( x^2 + b^2 = R^2 \) you should think of \( A = R \cos \theta \), \( B = R \sin \theta \).
So \( x = 2 \cos \theta \), \( y = 2 \sin \theta \) for \( 0 \leq \theta \leq 2\pi \),
and \( z = 1 - x - y = 1 - 2 \cos \theta - 2 \sin \theta \). Therefore
\[
\vec{r}(\theta) = (2 \cos \theta, 2 \sin \theta, 1 - 2 \cos \theta - 2 \sin \theta).
\]

Ex. Parametrize the curve of intersection of
\( x^2 + z^2 = 1 \), \( y^2 + z^2 = 1 \), and \( x \geq 0 \).

Solution. In these figures, the orange curve is what we are looking for. As you can see in Figure 2, for a given \( x, z \), there are two possible values for \( y \). But having \( y, z \), there is a unique \( x \). (For points
From front (through \( x \)-axis),
From left (through \( y \)-axis):
\[
\text{Figure 1, Figure 2, Figure 4, Figure 3.}
\]
on the orange curve.) So we start with y, z components:

\[ y^2 + z^2 = 1. \]

So let \( y = \cos \theta \), \( z = \sin \theta \) for \( 0 \leq \theta \leq 2\pi \) (as we can see in Figure 4, we have the full circle.) Since \( x^2 + z^2 = 1 \), we get

\[ x^2 = 1 - z^2 = 1 - \sin^2 \theta = \cos^2 \theta. \]

On the other hand, \( x \geq 0 \). Hence \( x = |\cos \theta| \).

Overall we get \( \vec{r}(\theta) = (|\cos \theta|, \cos \theta, \sin \theta) \)

for \( 0 \leq \theta \leq 2\pi \).

Ex. How can we visualize the curve \( \vec{r}(t) = (\cos t, \sin t, t) \)?

To visualize a curve given by a parametrization, one can try to find relations between its components. This way we might be able to find surfaces that contain the given curve.

Solution. We observe that \( x = \cos \theta \), \( y = \sin \theta \) satisfy \( x^2 + y^2 = 1 \). Hence this curve is part of this cylinder.
(Think about it as a bumblebee which is at \((\cos t, \sin t, t)\) after \(t\) seconds. So it is at \((1,0,0)\) at \(t=0\), and it flies upward, but its shadow on ground just rotates on a circle centered at the origin.) [It is called a helix.]

Now suppose in the above example, the temperature at any point is given \(T(x,y,z)\). We’d like to know what is the rate of change of temperature as the bumblebee flies away? More generally:

For a given vector-valued function \(\vec{r}(t) = (x(t), y(t))\) (it might have three components) and a given two-variable function \(f(x,y)\), how can we compute \(\frac{d}{dt} f(\vec{r}(t))\)?

By definition,

\[
\frac{d}{dt} f(\vec{r}(t)) = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}
\]

To understand this limit, we approximate \(f(x,y)\) by an affine function for \((x,y)\) close to \((x(t), y(t))\).
Lecture 16: Chain rule

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We know that for any \((a, b)\) we have

\[
 f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x-a) + \frac{\partial f}{\partial y}(a, b)(y-b)
\]

if \(f\) is differentiable at \((a, b)\) and \((x, y)\) is close to \((a, b)\).

Using this for \((a, b) = (x(t), y(t))\) and

\[
 (x, y) = (x(t+\Delta t), y(t+\Delta t)),
\]

we get:

\[
 f(x(t+\Delta t), y(t+\Delta t)) \approx 
\]

\[
 f(x(t), y(t)) + \frac{\partial f}{\partial x}(x(t), y(t))(x(t+\Delta t) - x(t)) + \frac{\partial f}{\partial y}(x(t), y(t))(y(t+\Delta t) - y(t))
\]

Hence

\[
 \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t} \approx 
\]

\[
 \frac{x(t+\Delta t) - x(t)}{\Delta t} \cdot \frac{\partial f}{\partial x}(x(t), y(t)) + \frac{y(t+\Delta t) - y(t)}{\Delta t} \cdot \frac{\partial f}{\partial y}(x(t), y(t))
\]

As \(\Delta t \to 0\), we get

\[
 \frac{x(t+\Delta t) - x(t)}{\Delta t} \to x'(t) \quad \text{and} \quad \frac{y(t+\Delta t) - y(t)}{\Delta t} \to y'(t)
\]

So we get

**Chain Rule**

\[
 \frac{df(r(t))}{dt} = \frac{\partial f}{\partial x}(r(t)) \cdot \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(r(t)) \cdot \frac{dy}{dt}(t)
\]

Sometimes we write:

\[
 \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
\]