In the previous lecture we discussed what a bounded region and a closed region are.

**Theorem** Any continuous function has a global max and a global min on a closed and bounded region.

**Question** How can we find global max and global min?

1. Find all the critical points of \( f \) inside \( D \).

2. Find max and min on the boundary of \( D \).

3. Compare the values of \( f \) at the critical pts in \( D \) and the max and min of \( f \) on the boundary of \( D \).

**Example** Find global max and global min of \( f(x,y) = x^2 + y^2 - x - y \) in the disk \( x^2 + y^2 \leq 1 \).

**Solution** Step 1. Find all critical points in \( x^2 + y^2 < 1 \).

\[ \nabla f = (2x-1, 2y-1) = (0, 0). \] So \( x = \frac{1}{2} \) and \( y = \frac{1}{2} \)

\[ \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} < 1. \] So \( \left(\frac{1}{2}, \frac{1}{2}\right) \) is a critical point in this region. And \( f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2} \).
Step 2. Find max and min of $f(x, y)$ constrained to the circle $x^2 + y^2 = 1$.

Let's parametrize the circle: $x = \cos \theta$, $y = \sin \theta$ and $0 \leq \theta \leq 2\pi$.

$$f(\cos \theta, \sin \theta) = \cos^2 \theta + \sin^2 \theta - \cos \theta - \sin \theta$$

$$= 1 - \cos \theta - \sin \theta$$

It is a single-variable function and you can use all the techniques from single-variable calculus:

Let $g(\theta) = 1 - \cos \theta - \sin \theta$. Then we have to find all $\theta$'s such that $g'(\theta) = 0$.

$$g'(\theta) = \sin \theta - \cos \theta = 0$$

So $\sin \theta = \cos \theta$. Therefore

$$\theta = \frac{\pi}{4} \quad \text{or} \quad \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$ 

Now we have to compare $g\left(\frac{\pi}{4}\right)$, $g\left(\frac{5\pi}{4}\right)$, $g(0)$, and $g(2\pi)$. We have $g\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 1 - \sqrt{2}$, $g\left(\frac{5\pi}{4}\right) = 1 + \sqrt{2}$, $g(0) = g(2\pi) = 0$. 
So maximum of $f$ on the circle occurs at $(\cos \frac{5\pi}{4}, \sin \frac{5\pi}{4})$

$= (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and it is $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 1+\sqrt{2}$,

and min. of $f$ on the circle occurs at $(\cos \frac{\pi}{4}, \sin \frac{\pi}{4})$

$= (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and it is $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 1-\sqrt{2}$.

**Step 3.** Comparing $f(\frac{1}{2}, \frac{1}{2})$, $1+\sqrt{2}$, and $1-\sqrt{2}$.

$f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$.

Clearly $1+\sqrt{2}$ is the largest value here.

$-\frac{1}{2} < 1-\sqrt{2} \iff \sqrt{2} < 1+\frac{1}{2} \iff \sqrt{2} < \frac{3}{2} \iff 2 < \frac{9}{4}$.

So global max $= f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 1+\sqrt{2}$.

and global min $= f(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{2}$.

For **step 2**, there are two methods. The first method (as we did above) is to parametrize the boundary of the given region. Then one can find the max and min of single-variable function $f(\vec{r}(t))$ where $\vec{r}(t)$ is a parametrization of (part of) boundary of the given region. This method is
particularly useful when the boundary consists of a few segments, e.g. it is a square or triangle.

The 2nd method is Lagrange multiplier method. This method is applicable when the boundary is a level curve (or a level surface).

**Lagrange multiplier method.**

**Problem.** Find max and min of $f(x,y)$ constrained to $g(x,y) = c_0$.

Let’s use the idea in the 1st method and assume $\mathbf{r}(t)$ is a parametrization of $g(x,y) = c_0$. So max and min of $f$ restricted to this curve occur at a critical point of the single-variable function $f(\mathbf{r}(t))$. This implies at the point $(x_0, y_0) = \mathbf{r}(t_0)$ we have

$$0 = \frac{d}{dt}(f(\mathbf{r}(t)))_{t=t_0} = \nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0).$$

And so $\nabla f(x_0, y_0) \perp \mathbf{r}'(t_0)$.

At the same time $\nabla g(x_0, y_0)$ is perpendicular to the
the level curve $g(x,y)=c_0$. So $\vec{F}'(t_0)$ (which is parallel to the tangent line of $g(x,y)=c_0$ at $(x_0,y_0)$) is perpendicular to $\nabla g(x_0,y_0)$.

Since both $\nabla f(x_0,y_0)$ and $\nabla g(x_0,y_0)$ are perpendicular to $\vec{F}'(t_0)$, we have

$$\nabla f(x_0,y_0) = c \nabla g(x_0,y_0)$$

(A similar argument works for 3 variable functions.)

Because of the multiplier $c$, it is called Lagrange multiplier method (the reason for the Lagrange part of this name should be clear.)