1. (5 points) Determine if the lines given by the vector parametrizations

$$\mathbf{r}_1(t) = \langle 5, -16, 19\rangle + t\langle 1, -3, 4\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 5, -1, -11\rangle + t\langle -2, 1, 2\rangle$$

intersect and, if so, find the point of intersection.

If they intersect at $r_1(t_1) = r_2(t_2)$, then we have

$$\begin{cases} 5 + t_1 = 5 - 2t_2 & (1) \\ -16 - 3t_1 = -1 + t_2 & (2) \\ 19 + 4t_1 = -11 + 2t_2 & (3) \end{cases}$$

$\overset{(1)}{\Rightarrow} t_1 = -2t_2 \overset{(2)}{\Rightarrow} -16 + 6t_2 = -1 + t_2$

$\Rightarrow 5t_2 = 15$

$\Rightarrow t_2 = 3 \Rightarrow t_1 = -6$

Let's check if they satisfy (3).

$19 + (4)(-6) \overset{?}{=} -11 + (2)(3)$

The left hand side $= 19 - 24 = -5$

The right hand side $= -11 + 6 = -5$

$\Rightarrow$ these Lines intersect at $r_1(-6) = r_2(3)$

$= \langle -1, 2, -5\rangle$

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2. (5 points) Find a vector parametrization of the line tangent to the curve

$$\mathbf{r}(t) = \langle t^3, 2t, t^2 + 1\rangle$$

at the point $(8, 4, 5)$.

- A vector parametrization of the line tangent to the curve $r(t)$ at $r(t_0)$ is $L(t) = r(t_0) + t\,r'(t_0)$.
- $r(t) = \langle 8, 4, 5\rangle$ implies $t = 2$.
- $r'(t) = \langle 3t^2, 2, 2t\rangle$

So $L(t) = r(2) + t\,r'(2)$

$= \langle 8, 4, 5\rangle + t\langle 12, 2, 4\rangle$

3. (5 points) Give a vector parametrization of the circle contained in the plane $x = 2$ with radius 3 and center at the point $(2, 1, 3)$.

A vector parametrization of the circle contained in the $x=0$ (the yz-plane) with radius 3 and center at the origin is $\langle 0, 3\cos\theta, 3\sin\theta \rangle$ for $0 \le \theta < 2\pi$.

So a vector parametrization of the given circle is

$$\vec{r}(\theta) = \langle 2, 1, 3 \rangle + \langle 0, 3\cos\theta, 3\sin\theta \rangle$$

$$= \langle 2, 1 + 3\cos\theta, 3 + 3\sin\theta \rangle.$$

[NOT in our exam 1]

4. (5 points) Compute the length of the helix parameterized by

$$\mathbf{r}(t) = \langle \cos t, 3t, \sin t \rangle$$

over the interval $t = 0$ to $t = 2\pi$.

$$\int_0^{2\pi} \|r'(t)\|\, dt = \int_0^{2\pi} \|\langle -\sin t, 3, \cos t \rangle\|\, dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + 9 + \cos^2 t}\; dt$$

$$= \int_0^{2\pi} \sqrt{10}\; dt$$

$$= 2\sqrt{10}\,\pi .$$

5. (5 points) Find an equation for the plane (in the form $ax + by + cz = d$) that contains both of the parallel lines given by the vector parameterizations

$$\mathbf{r}_1(t) = \langle 1, 2, 3\rangle + t\langle 1, -1, 1\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 0, 1, 5\rangle + t\langle 1, -1, 1\rangle$$

- Since it contains $r_1(t)$, it is parallel to $\langle 1,-1,1\rangle$.
- Since it contains $r_1(t)$ and $r_2(t)$, it contains both points $P=(1,2,3)$ and $Q=(0,1,5)$. So it is parallel to $\vec{PQ} = \langle 0-1, 1-2, 5-3\rangle = \langle -1,-1,2\rangle$. Hence $\vec{n} = \langle 1,-1,1\rangle \times \langle -1,-1,2\rangle$ is a normal vector.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -1 & -1 & 2 \end{vmatrix} = \left\langle \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} \right\rangle$$

$$= \langle -1, -3, -2\rangle$$

Since this plane contains P,

$$-x - 3y - 2z = -1 - 6 - 6 = -13.$$

$$\Rightarrow x + 3y + 2z = 13.$$

6. (1 point each) *No work or justification is needed for credit.*

(a) Suppose $\mathbf{w} \times \mathbf{v} = \langle 1, 2, 3 \rangle$. What does $\mathbf{v} \times (3\mathbf{w})$ or $\mathbf{v} \times (2\mathbf{w})$ equal?

$v \times (3w) = 3\, v \times w = -3\, w \times v = \langle -3, -6, -9 \rangle$

$v \times (2w) = 2\, v \times w = -2\, w \times v = \langle -2, -4, -6 \rangle$

(b) What can you say about the angle between $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a} \cdot \mathbf{b} < 0$?

It is obtuse. Recall. $\cos\theta = \dfrac{a \cdot b}{\|a\| \, \|b\|}$. So
$\begin{cases} \theta : \text{acute} & \text{if } a \cdot b > 0 \\ \theta : \text{obtuse} & \text{if } a \cdot b < 0 \\ \theta = \pi/2 & \text{if } a \cdot b = 0 \end{cases}$

(c) To which coordinate plane (xy-, xz- or yz-plane) is the plane $x = -3$ or $y = -3$ parallel?

$x = -3$ is parallel to $x = 0$ which is the yz-plane. $\{$ $y = -3$ is parallel to xz-plane

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(d) Suppose that $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \langle 1, 2, 3 \rangle$. What is $\text{proj}_{2\mathbf{v}}(3\mathbf{u})$ or $\text{proj}_{3\mathbf{v}}(2\mathbf{u})$?

(e) Answer true or false. If $\mathbf{v} \neq \mathbf{0}$, then $-\dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector.

True. Recall. $\|c\vec{v}\| = |c| \, \|\vec{v}\|$. So

$\left\| \dfrac{-1}{\|v\|} v \right\| = \left| \dfrac{-1}{\|v\|} \right| \|v\| = \dfrac{1}{\|v\|} \|v\| = 1.$

$\text{Proj}_v(u) = \dfrac{u \cdot v}{v \cdot v} v$. So $\text{Proj}_{2v}(3u) = \dfrac{(3u)\cdot(2v)}{(2v)\cdot(2v)} 2v$

$\Rightarrow \text{Proj}_{2v}(3u) = \dfrac{6}{4} \dfrac{u \cdot v}{v \cdot v} 2v = 3 \dfrac{u \cdot v}{v \cdot v} v = 3\,\text{Proj}_v(u) = \langle 3, 6, 9 \rangle.$

$\text{Proj}_{3v}(2u) = \dfrac{(2u)\cdot(3v)}{(3v)\cdot(3v)} 3v = 2 \dfrac{u \cdot v}{v \cdot v} v = 2\,\text{Proj}_v(u) = \langle 2, 4, 6 \rangle.$