Math 21C Midterm II, Fall 02, Lindblad.

1. A particle moves with position vector given by \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t^{3/2} \mathbf{k} \).
   (a) Find the velocity and speed of the particle at time \( t = \pi \).
   (b) How far does the particle travel between time \( t = 0 \) and \( t = 10 \)?

2. Let \( F(x, y) = x^3 - x^2 + y^2 - y + 1 \)
   (a) Find the gradient \( \nabla F(x, y) \).
   (b) Find the directional derivative in the direction of \( \langle 1, 2 \rangle \) at the point \( (2, 3) \).
   (c) Let \( \mathbf{r}(t) = \langle f(t), g(t) \rangle \) be a curve such that \( \mathbf{r}(0) = \langle 1, 0 \rangle \) and \( \mathbf{r}'(0) = \langle 1, 1 \rangle \). Let \( h(t) = F(\mathbf{r}(t)) \). Find \( h'(0) \).

3. Consider the surface given by \( z = f(x, y) = \sqrt{x^2 + 3y^2} \).
   a) Find the tangent plane to the surface at the point \( (1, 1, 2) \).
   b) A student was asked to find an approximation for \( f(1.1, 1.2) \) but the professor did not allow calculators. The student noticed that \( f(1.1, 1.2) \) is approximately \( f(1, 1) = \sqrt{1 + 3} = 2 \). Use the linear approximation to get a better approximation.

4. Let \( f(x, y) = x^3 - x^2 + y^2 - y + 1 \). Find the critical points of \( f(x, y) \) and determine if they are local max, min or saddle points. Are there any absolute max or min?

5. Use Lagrange multipliers to find the point on the hyperboloid \( \{(x, y, z); z^2 = x^2 + y^2 + 1, z \geq 0\} \) that is closest to the point \( (0, 0, -2) \). (NOT PART OF OUR EXAM)
1. (a) \( \mathbf{r}(t) = \langle \cos t, \sin t, t^{3/2} \rangle \)

\[ \mathbf{v}(t) = \mathbf{r}'(t) = \langle -\sin t, \cos t, \frac{3}{2} t^{1/2} \rangle \]

\[ s(t) = \| \mathbf{v}(t) \| = \sqrt{(-\sin t)^2 + \cos^2 t + \left(\frac{3}{2} t^{1/2}\right)^2} \]

\[ = \sqrt{1 + \frac{9}{4} t} \]

\[ \Rightarrow \mathbf{v}(\pi) = \langle 0, -1, \frac{3}{2} \sqrt{\pi} \rangle \]

\[ s(\pi) = \sqrt{1 + \frac{9\pi}{4}} \]

(b) **Total length**

\[ = \int_a^b \| \mathbf{r}'(t) \| \, dt \]

\[ = \int_0^{10} \sqrt{1 + \frac{9}{4} t} \, dt \]

\[ = \int_1^{45/2} \left(\sqrt{u}\right)\left(\frac{4}{9}\right) du \]

\[ = \frac{4}{9} \cdot \frac{2}{3} \left. u^{3/2} \right|_1^{45/2} \]

\[ = \frac{8}{27} \left( \frac{45}{2} \sqrt{\frac{45}{2}} - 1 \right) \]

\[ = \frac{20}{3} \sqrt{\frac{45}{2}} - \frac{8}{27} \]
2. (a) \( F(x,y) = x^3 - x^2 + y^2 - y + 1 \)
\[ \nabla F = \langle 3x^2 - 2x, 2y - 1 \rangle \]

(b) \( D_u F = \vec{u} \cdot \nabla F \) where \( \vec{u} = \frac{\nabla}{\|\nabla\|} \)

\[ \Rightarrow \vec{u} = \frac{\langle 1, 2 \rangle}{\sqrt{1+4}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle. \]

\( \nabla F(2,3) = \langle 8, 5 \rangle \)

\[ \Rightarrow D_u F(2,3) = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \cdot \langle 8, 5 \rangle \]

\[ = \frac{18}{\sqrt{5}} = \frac{18\sqrt{5}}{5} \]

(c) \( \vec{r}'(0) = \langle 1, 0 \rangle \) and \( \vec{r}'(0) = \langle 1, 1 \rangle \).

\[ \frac{d}{dt} F(\vec{r}'(t)) \bigg|_{t=0} = \vec{r}'(0) \cdot \nabla F(\vec{r}(0)) \]

\[ = \langle 1, 1 \rangle \cdot \nabla F(1,0) \]

\[ = \langle 1, 1 \rangle \cdot \langle 1, -1 \rangle = 0. \]
3. \( z = f(x, y) = \sqrt{x^2 + 3y^2} \).

(a) Tangent plane at \((1, 1, 2)\)

\[
\nabla f = \frac{1}{2\sqrt{x^2 + 3y^2}} \left< 2x, 6y \right> = \frac{1}{\sqrt{x^2 + 3y^2}} \left< x, 3y \right>
\]

\( \Rightarrow \nabla f(1, 1) = \frac{1}{2} \left< 1, 3 \right> \).

\( \Rightarrow \ z = 2 + \frac{1}{2} (x - 1) + \frac{3}{2} (y - 1) \)

\( \Rightarrow \ z = \frac{x}{2} + \frac{3}{2} y \)

(b) \( f(1.1, 1.2) \approx ? \)

\[
f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y
\]

\[
f(1.1, 1.2) \approx 2 + \frac{1}{2} (0.1) + \frac{3}{2} (0.2)
\]

\[
= 2 + 0.05 + 0.3
\]

\( = 2.35 \)
4. \( f(x,y) = x^3 - x^2 + y^2 - y + 1 \)

\[ \nabla f = \langle 3x^2 - 2x, 2y - 1 \rangle = \langle 0, 0 \rangle \]

\[ \Rightarrow \begin{cases} 3x^2 - 2x = 0 & \Rightarrow x = 0 \text{ or } x = \frac{2}{3} \\ 2y - 1 = 0 & \Rightarrow y = \frac{1}{2} \end{cases} \]

\[ \Rightarrow \text{critical pts are} \ (0, \frac{1}{2}) \text{ and} \ (\frac{2}{3}, \frac{1}{2}) \cdot \]

\[ f_{xx} = 6x - 2 \quad \Rightarrow \quad D = f_{xx} f_{yy} - f_{xy}^2 \]

\[ f_{xy} = 0 \quad \Rightarrow \quad D = (6x - 2)(2) = 4(3x - 1) \cdot \]

\[ f_{yy} = 2 \]

<table>
<thead>
<tr>
<th>Critical Pts</th>
<th>( D )</th>
<th>( f_{xx} )</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, \frac{1}{2}))</td>
<td>-</td>
<td></td>
<td>\text{saddle pt}</td>
</tr>
<tr>
<td>((\frac{2}{3}, \frac{1}{2}))</td>
<td>+</td>
<td>+</td>
<td>\text{local min}</td>
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</tbody>
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No, there is neither an absolute max nor an absolute min. Since \( f(x,0) = x^3 - x^2 + 1 \) is an degree polynomial, the range of \( f \) is \((-\infty, +\infty)\). So there is no abs. max or abs. min.
5. \( z^2 = x^2 + y^2 + 1 \), \( z \geq 0 \)

Closest to \((0,0,-2)\)

\[ \Rightarrow \min \text{ of } f(x,y,z) = x^2 + y^2 + (z+2)^2 \]

under the condition:

\[ g(x,y,z) = x^2 + y^2 - z^2 = -1 \]

and \( z \geq 0 \).

We have to solve the following system of equations:

\[
\begin{align*}
\nabla f &= c \nabla g \\
g &= -1 \\
z &\geq 0
\end{align*}
\]

\[ \Rightarrow \langle 2x, 2y, 2(z+1) \rangle = c \langle 2x, 2y, -2z \rangle \]

\[ \Rightarrow \begin{cases} 
2x = 2cx \\
2y = 2cy \\
2z + 2 = -2cz 
\end{cases} \Rightarrow \begin{cases} 
x = 0 \quad \text{or} \quad c = 1 \\
y = 0 \quad \text{or} \quad c = 1 \\
z = -\frac{1}{2}
\end{cases} \]

If \( c = 1 \), then \( 2z + 2 = -2z \Rightarrow z = -\frac{1}{2} \)

Not possible as \( z \geq 0 \).

So \( c \neq 1 \). Hence \( x = y = 0 \) \( \Rightarrow \) \( z^2 = 1 \) \( \Rightarrow \) \( z = 1 \)

\[ x^2 + y^2 - z^2 = -1 \int z \geq 0 \]

\[ \Rightarrow (0,0,1) \text{ is the closest pt to } (0,0,-2). \]