Summary of the first week's lectures.

① Basics of 2D and 3D vectors.

a $\vec{v} = \langle x, y, z \rangle$ (Components of a vector)

b $c\vec{v} = \langle cx, cy, cz \rangle$ (Scalar multiplication)

c $\vec{v} + \vec{w} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$ where

$\vec{v} = \langle x_1, y_1, z_1 \rangle$ and $\vec{w} = \langle x_2, y_2, z_2 \rangle$.

d $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$ (length of $\vec{v}$)

e $\|c\vec{v}\| = |c| \, \|\vec{v}\|$.

f We say $\vec{v}$ and $\vec{w}$ are parallel if

$$\vec{w} = c\vec{v}$$

for some non-zero number c.

g We say $\vec{v}$ and $\vec{w}$ have the same direction (or $\vec{w}$ is in the direction of $\vec{v}$) if

$$\vec{w} = c\vec{v}$$

for some positive number c.

h $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ where

$P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$.

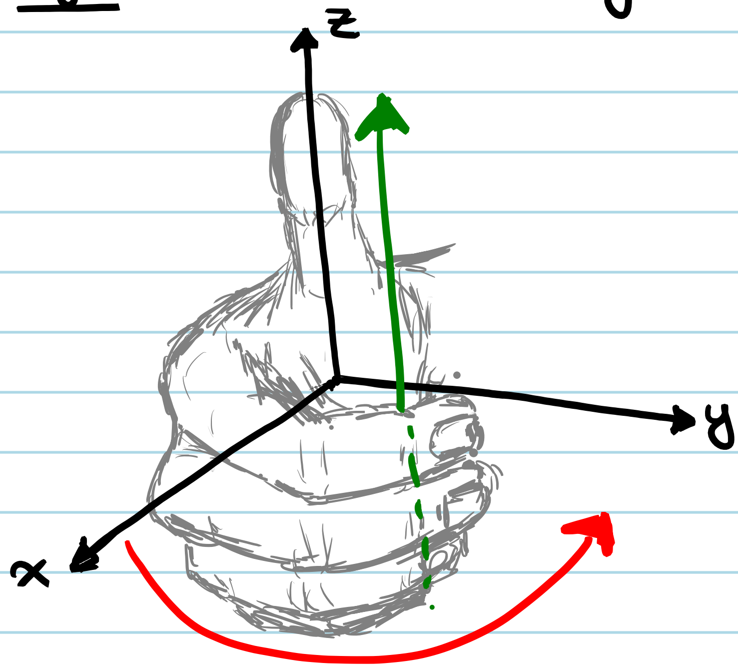
$\boxed{i}$ $\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$ and $\vec{k} = \langle 0,0,1\rangle$

(Standard basis)

$\boxed{j}$ $\langle x, y, z\rangle = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

<u>**Warning**</u>. Please do NOT write $\langle x\vec{i}, y\vec{j}, z\vec{k}\rangle$ or $\langle x+y+z\rangle$ instead of $\langle x, y, z\rangle$ or $x\vec{i} + y\vec{j} + z\vec{k}$.

$\boxed{k}$ The <u>xyz</u> axis and the right-hand rule.



(2) Find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

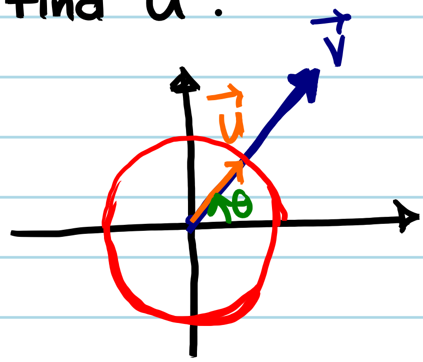
Solution : $\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}$.

(3) In 2D, we can use the angle that $\vec{v}$ makes with the positive x-axis to find $\vec{u}$.

$\vec{v} = \|\vec{v}\| \langle \cos\theta, \sin\theta \rangle$

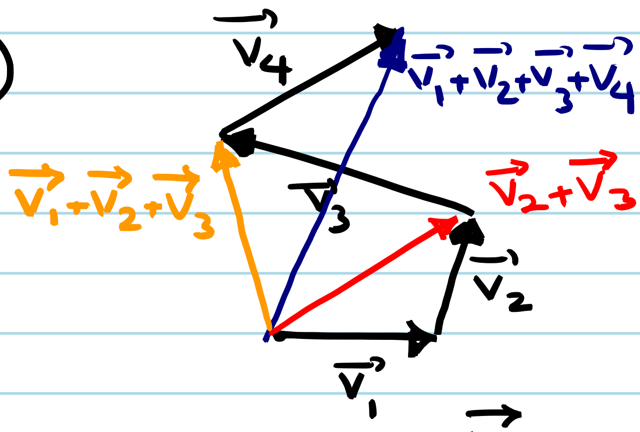
and

$\vec{u} = \langle \cos\theta, \sin\theta \rangle$.

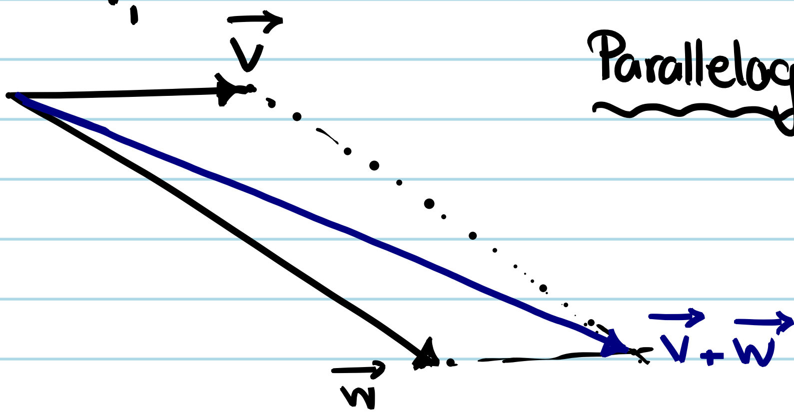


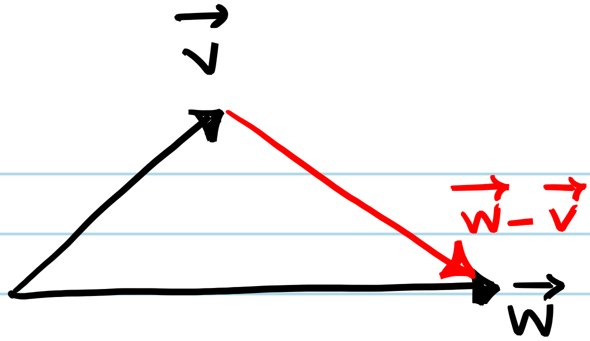
$\boxed{\text{Warning}}$ To find the angle θ, the initial point of the vector should be at the origin.

(4) How to visualize sum of vectors.

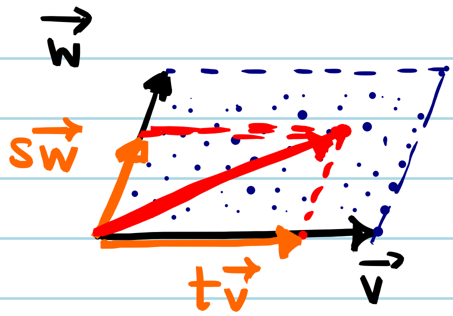


Parallelogram law



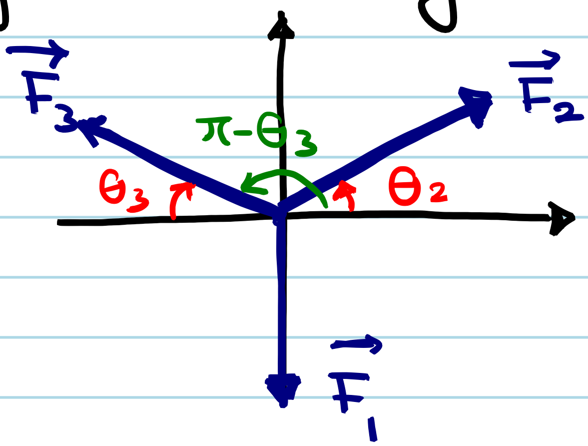


⑤ The parallelogram spanned by two vectors:



$$\{t\vec{v}+s\vec{w} \mid 0 \le s,t \le 1\}$$

⑥ Example from physics. After getting to the math context, we have three vectors $\vec{F}_1, \vec{F}_2, \vec{F}_3$; we are usually in the following situation:



We know $\|\vec{F}_1\|$, θ_2 and θ_3. We also know

$$\vec{F}_1+\vec{F}_2+\vec{F}_3=\vec{0}.$$

And we are asked to find $\|\vec{F}_2\|$ and $\|\vec{F}_3\|$ (or $\vec{F}_2$ and $\vec{F}_3$).

Outline of solution. Let $\|\vec{F_1}\| = f_1$, $\|\vec{F_2}\| = f_2$ and $\|\vec{F_3}\| = f_3$. Then

- $\vec{F_1} = \langle 0, -f_1 \rangle$
- $\vec{F_2} = \|\vec{F_2}\| \langle \cos\theta_2, \sin\theta_2 \rangle = f_2 \langle \cos\theta_2, \sin\theta_2 \rangle$
- $\vec{F_3} = f_3 \langle \cos(\pi-\theta_3), \sin(\pi-\theta_3) \rangle$

$$= f_3 \langle -\cos\theta_3, \sin\theta_3 \rangle .$$

Therefore $\vec{0} = \vec{F_1} + \vec{F_2} + \vec{F_3}$

$$= \langle 0, -f_1 \rangle + \langle (\cos\theta_2) f_2, (\sin\theta_2) f_2 \rangle + \langle -(\cos\theta_3) f_3, (\sin\theta_3) f_3 \rangle$$

$$= \langle (\cos\theta_2) f_2 - (\cos\theta_3) f_3, -f_1 + (\sin\theta_2) f_2 + (\sin\theta_3) f_3 \rangle .$$

Hence
$$\begin{cases} (\cos\theta_2) f_2 - (\cos\theta_3) f_3 = 0 \\ (\sin\theta_2) f_2 + (\sin\theta_3) f_3 = f_1 \end{cases}$$

And we solve these equations for f_2 and f_3.

7 Equations and their description.

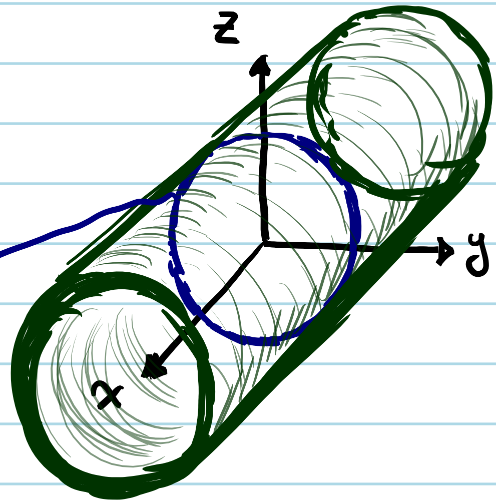
a Equation of a sphere of radius R centered at (a,b,c) is

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2.$$

b Cylindrical Surfaces.

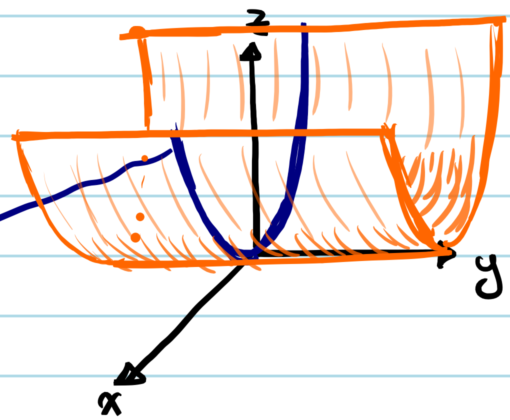
(i) $y^2+z^2=1$.

a circle in the yz-plane



(ii) $z=x^2$.

a parabola in the xz-plane



a′ We also saw these examples:

$$x^2+y^2+z^2=z \Rightarrow x^2+y^2+(z^2-z+1/4)=1/4$$
$$\Rightarrow x^2+y^2+(z-1/2)^2=(1/2)^2$$

Sphere of radius $1/2$ centered at $(0,0,1/2)$.

C Equation of a line

(i) Passing through $P_0 = (x_0, y_0, z_0)$ and parallel to $\vec{v} = \langle a, b, c \rangle$:

Vector Parametrization

$$\overrightarrow{r(t)} = \overrightarrow{OP_0} + t\,\vec{v}$$

$$= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

$$= \langle at + x_0, bt + y_0, ct + z_0 \rangle.$$

Parametric Equations

$$x = at + x_0, \quad y = bt + y_0, \quad z = ct + z_0.$$

(ii) Passing throug two points

$$P = (x_1, y_1, z_1) \text{ and } Q = (x_2, y_2, z_2).$$

Vector Parametrization

$$\overrightarrow{r(t)} = t\,\overrightarrow{OQ} + (1-t)\,\overrightarrow{OP}$$

$$= \langle x_1 + (x_2 - x_1)t, y_1 + (y_2 - y_1)t, z_1 + (z_2 - z_1)t \rangle$$

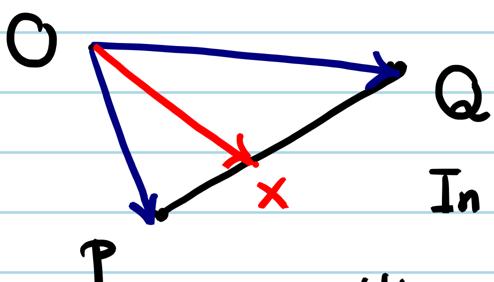
Parametric Equations

$$x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t, \quad z = z_1 + (z_2 - z_1)t$$

$\boxed{d}$ **Parametrizing a segment** PQ where $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$ (from P toward Q).

$$\overrightarrow{r(t)} = t\,\overrightarrow{OQ} + (1-t)\,\overrightarrow{OP}, \quad 0 \le t \le 1.$$

$\boxed{e}$ **Points on a segment.**



$$\overrightarrow{OX} = \frac{|PX|}{|PQ|}\,\overrightarrow{OQ} + \frac{|QX|}{|PQ|}\,\overrightarrow{OP}.$$

In particular, if M is the middle point of PQ, then

$$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right).$$