Summary of the third week's lectures

How to find equation of a plane

Exp. 1. [Given: a normal vector $\langle a, b, c\rangle$ and a point (x_0, y_0, z_0).

Equation: $ax + by + cz = ax_0 + by_0 + cz_0$.]

Find an equation of a plane containing $(1,2,3)$ with a normal vector $\langle 1,-1,1\rangle$.

Solution. $(1)x + (-1)y + (1)z = (1)(1) + (-1)(2) + (1)(3)$

So $x - y + z = 2$.

Exp. 2 [Given: three points P_1, P_2 and P_3.

$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is a normal vector. Now we can use **Exp. 1** for $\vec{n}$ and P_1.]

Find an equation of a plane which contains

$P_1 = (1,0,1)$, $P_2 = (0,1,1)$ and $P_3 = (1,1,0)$.

Solution. $\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is a normal vector.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \right\rangle$$

$$= \langle -1, -1, -1\rangle.$$

So $-x-y-z=-1-0-1=-2$ is an equation of this plane.

Exp. 3 [Given: Two parallel vectors $\vec{v}$ and $\vec{w}$ and a point (x_0, y_0, z_0).

$\vec{n}=\vec{v}\times\vec{w}$ is a normal vector. Now we can use Exp. 1.]

Exp. 4 [Given: Contains two intersecting lines:

$$\vec{r_1}(t)=\overrightarrow{OP_0}+t\,\vec{v_1}$$

and $$\vec{r_2}(t)=\overrightarrow{OP_0}+t\,\vec{v_2}.$$

Then $\vec{n}=\vec{v_1}\times\vec{v_2}$ is a normal vector and again we can use Exp. 1.]

Exp. 5 [Given: A parallel plane: $ax+by+cz=d$
A point: (x_0, y_0, z_0)

Two parallel planes share normal vectors. So $\langle a,b,c\rangle$ is a normal vector of the considered plane. Hence an equation of this plane is

$$ax+by+cz=ax_0+by_0+cz_0.]$$

Exp. 6 [Given: Contains two parallel lines

$$\vec{r}_1(t) = \overrightarrow{OP_1} + t\,\vec{v}$$
and $$\vec{r}_2(t) = \overrightarrow{OP_2} + t\,\vec{v}.$$

Then $\vec{n} = \overrightarrow{P_1P_2} \times \vec{v}$ is a normal vector. Now we can use Exp 1 for $\vec{n}$ and the point P_1.]

Vector-valued functions

A function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is called a vector-valued function (for now).

This type of function is helpful to

- Parametrize a curve
- Describe the position vector of a moving particle.
- Find velocity or acceleration of a moving particle.
- etc.

Exp 1. Parametrize the segment PQ where

$$P = (1, 0\ 1) \quad \text{and} \quad Q = (2, 3, 4).$$

Solution. $\vec{r}(t) = t\,\overrightarrow{OQ} + (1-t)\,\overrightarrow{OP}$ for $0 \le t \le 1$.

$$\vec{r}(t) = t\langle 2,3,4\rangle + (1-t)\langle 1,0,1\rangle$$
$$= \langle 1+t, 3t, 1+3t\rangle \qquad \text{for } 0 \le t \le 1.$$

Exp 2. Parametrize the line through $P_0 = (1,2,3)$ and parallel to $\vec{v} = \langle 4,5,6 \rangle$.

Solution. $\vec{r}(t) = \overrightarrow{OP_0} + t\,\vec{v}$

$$= \langle 1,2,3 \rangle + t\langle 4,5,6 \rangle$$

$$= \langle 1+4t,\ 2+5t,\ 3+6t \rangle.$$

Exp 3. Parametrize the circle of radius 2, centered at the origin in the yz-plane.

Solution. $x = 0$ (Since in the yz-plane)

$y = 2\cos\theta$ and $z = 2\sin\theta$ for $0 \le \theta < 2\pi$.

So $\vec{r}(\theta) = \langle 0,\ 2\cos\theta,\ 2\sin\theta \rangle$ for $0 \le \theta < 2\pi$.

Remark. Whenever you see $A^2 + B^2 = 1$, you should think of $A = \cos\theta$ and $B = \sin\theta$!

Exp 4. Parametrize an ellipse with center $(1,-2,-1)$ which is a translation of the ellipse

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

in the xy-plane.

Solution. A parametrization of the ellipse

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$$

in the xy-plane is

$$\left.\begin{array}{l}\frac{x}{2}=\cos\theta\\ \frac{y}{3}=\sin\theta\\ z=0\end{array}\right\} \Rightarrow \quad \langle 2\cos\theta, 3\sin\theta, 0\rangle.$$

So a parametrization of its translate with center

$$(1,-2,-1)$$

is $\vec{r}(\theta)=\langle 1,-2,-1\rangle+\langle 2\cos\theta, 3\sin\theta, 0\rangle$

$$=\langle 1+2\cos\theta, -2+3\sin\theta, -1\rangle$$

for $0\le\theta<2\pi$.

Exp 5. Parametrize the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$.

Solution. $x=\cos\theta$, $y=\sin\theta$, for $0\le\theta<2\pi$.

$$z=1-x-y=1-\cos\theta-\sin\theta.$$

So $\vec{r}(\theta)=\langle\cos\theta, \sin\theta, 1-\cos\theta-\sin\theta\rangle$ for $0\le\theta<2\pi$.

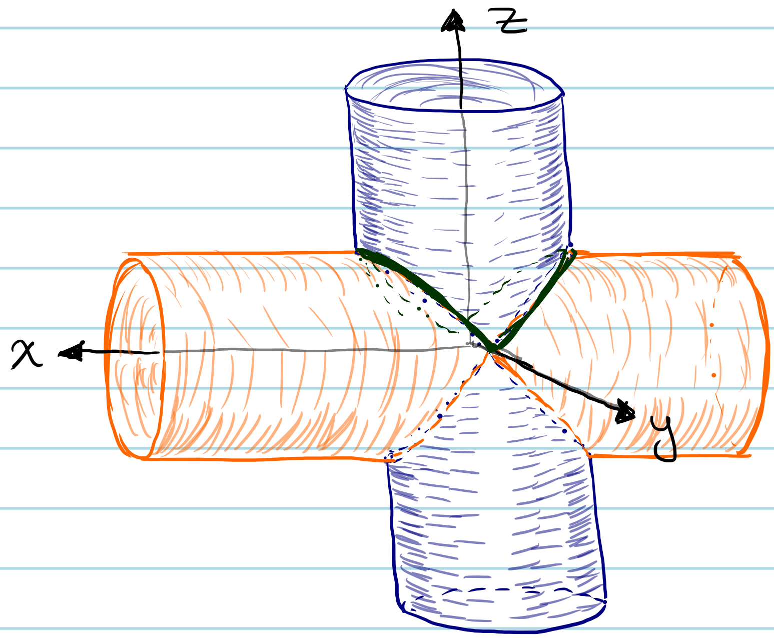
Exp 6. Parametrize the curve of intersection of the cylinders $x^2+y^2=1$ and $y^2+z^2=1$, and $z \geq 0$.

Solution. $x = \cos\theta$, $y = \sin\theta$, for $0 \leq \theta < 2\pi$.

$$z^2 = 1-y^2 = 1-\sin^2\theta = \cos^2\theta$$

Since $z \geq 0$, $z = |\cos\theta|$. So

$$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, |\cos\theta| \rangle .$$



Calculus of vector-valued functions

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle$$

Exp. Find $\lim_{t\to 0} \langle \frac{e^t-1}{t}, \frac{\sin t}{t}, \cos t \rangle$.

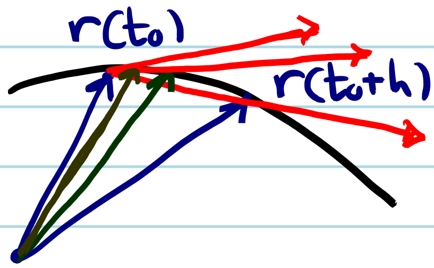
Solution. $\lim_{t\to 0} \langle \frac{e^t-1}{t}, \frac{\sin t}{t}, \cos t \rangle =$

$$\left\langle \lim_{t\to 0} \frac{e^t - 1}{t}, \lim_{t\to 0} \frac{\sin t}{t}, \lim_{t\to 0} \cos t \right\rangle =$$

(L'Hopital's rule) $$\left\langle \lim_{t\to 0} \frac{e^t}{1}, \lim_{t\to 0} \frac{\cos t}{1}, \cos(0) \right\rangle =$$

$$\langle 1, 1, 1 \rangle.$$

Derivative $\vec{r}\,'(t_0) = \lim_{h\to 0} \dfrac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h}$



So $\vec{r}\,'(t_0)$ is parallel to the tangent line of $\vec{r}(t)$ at $\vec{r}(t_0)$.

Also we have $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$.

<u>Exp</u>. [Parametrization of the tangent line of $\vec{r}(t)$ at $\vec{r}(t_0)$:

$$\vec{L}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)]$$

Find a parametrization of the tangent line of a helix $\vec{r}(t) = \langle \cos(2t), t, \sin(2t) \rangle$ at

$\vec{r}(\pi/4) = \langle 0, \pi/4, 1 \rangle$.

<u>Solution</u>. $L(t) = r(\pi/4) + t\, r'(\pi/4)$.

$r'(t) = \langle -2\sin(2t), 1, 2\cos(2t) \rangle$.

So $L(t) = \langle 0, \pi/4, 1 \rangle + t\langle -2, 1, 0 \rangle$

$$= \langle -2t, t + \pi/4, 1 \rangle.$$

A moving particle

If $\vec{r}(t)$ is the position vector of a moving particle, then

its velocity $\vec{v}(t) = \vec{r}\,'(t)$.

its acceleration $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

its speed $s(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$.

<u>Exp</u>. The position vector of a moving particle is

$$\vec{r}(t) = \langle \ln t, t+1, t^2 \rangle$$

(a) Find its velocity at $t=1$.

(b) Find its acceleration at $t=1$.

(c) Find its speed at $t=1$.

Solution. $r'(t) = \langle \frac{1}{t}, 1, 2t \rangle$

$r''(t) = \langle \frac{-1}{t^2}, 0, 2 \rangle$

(a) $r'(1) = \langle 1, 1, 2 \rangle$

(b) $r''(1) = \langle -1, 0, 2 \rangle$

(c) $\|r'(1)\| = \sqrt{1+1+4} = \sqrt{6}$.

Warning. Non-zero acceleration does NOT mean that the speed is not constant. It only means that the velocity is not constant.

Exp. The position vector of a moving particle is

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

(a) Find its speed.

(b) Find its acceleration.

Solution. (a) $s(t) = \|r'(t)\| = \|\langle -\sin t, \cos t, 1 \rangle\|$

$$= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

(b) $a(t) = r''(t) = \langle -\cos t, -\sin t, 0 \rangle$

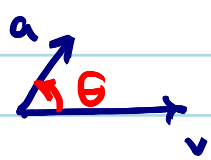
$$\boxed{r'(t) = \langle -\sin t, \cos t, 1 \rangle}$$

Notice In the above example, speed is constant, but acceleration is NOT zero.

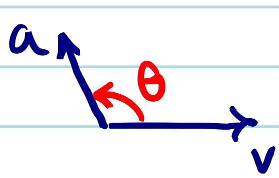
$\boxed{Q}$ How can we determine if a moving particle is speeding up or slowing down at $t = t_0$?

Intution
- If $\text{Proj}_{\vec{v}}\, \vec{a}$ and $\vec{v}$ have the same direction, then it is speeding up.
- If $\text{Proj}_{\vec{v}}\, \vec{a}$ and $\vec{v}$ have opposite directions, then it is slowing down.
- If $\text{Proj}_{\vec{v}}\, \vec{a} = 0$, then the speed is constant.

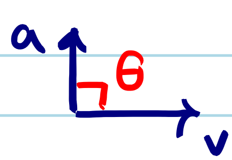
Geometrically



acute $\Rightarrow$ Speeding up



obtuse $\Rightarrow$ slowing down



$\theta = \pi/2 \Rightarrow$ Constant speed.

Computational $\vec{a} \cdot \vec{v} > 0 \Rightarrow$ Speeding up

$\vec{a}\cdot\vec{v} < 0 \Rightarrow$ Slowing down

$\vec{a}\cdot\vec{v} = 0 \Rightarrow$ Constant speed

Mathematical reasoning

It is speeding up at $t = t_0$ if and only if $s'(t_0) > 0$ where $s(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$ is the speed function.

$$s'(t) = \frac{d}{dt}\sqrt{\vec{v}(t)\cdot\vec{v}(t)}$$

(Chain rule)

$$= \frac{1}{2\sqrt{\vec{v}(t)\cdot\vec{v}(t)}}\,\frac{d}{dt}\left(\vec{v}(t)\cdot\vec{v}(t)\right)$$

(Dot product rule)

$$= \frac{1}{2\,\|\vec{v}(t)\|}\left(\vec{v}\,'(t)\cdot\vec{v}(t) + \vec{v}(t)\cdot\vec{v}\,'(t)\right)$$

$$= \frac{2\,\vec{a}(t)\cdot\vec{v}(t)}{2\,\|\vec{v}(t)\|} = \frac{\vec{a}(t)\cdot\vec{v}(t)}{\|\vec{v}(t)\|}.$$

(This is actually the component of $\vec{a}$ along $\vec{v}$.)

The total distance traveled over $a \le t \le b$ is

$$\int_a^b \|\vec{r}\,'(t)\|\,dt.$$