Summary of the third week's lectures

How to find equation of a plane.

Exp. 1. [Given: a normal vector $\langle a, b, c \rangle$ and a point $(x_0, y_0, z_0)$.]

Equation: $ax + by + cz = ax_0 + by_0 + cz_0$.

Find an equation of a plane containing $(1, 2, 3)$ with a normal vector $\langle 1, -1, 1 \rangle$.

Solution. $(1) \ x + (-1) y + (1) z = (1)(1) + (-1)(2) + (1)(3)$

So $x - y + z = 2$.

Exp. 2. [Given: three points $P_1, P_2$ and $P_3$.]

$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is a normal vector. Now we can use Exp. 1 for $\vec{n}$ and $P_1$.

Find an equation of a plane which contains $P_1 = (1, 0, 1)$, $P_2 = (0, 1, 1)$ and $P_3 = (1, 1, 0)$.

Solution. $\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is a normal vector.

$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} \rangle = \langle -1, -1, 1 \rangle$.
So \(-x-y-z = -1-0-1 = -2\) is an equation of this plane.

**Exp. 3** [Given: Two parallel vectors \(\vec{v}\) and \(\vec{w}\) and a point \((x_0, y_0, z_0)\).

\[\vec{n} = \vec{v} \times \vec{w}\] is a normal vector. Now we can use **Exp. 1**.]

**Exp. 4** [Given: Contains two intersecting lines:

\[\vec{r}_1(t) = \vec{r}_0 + t \vec{v}_1\]
and \[\vec{r}_2(t) = \vec{r}_0 + t \vec{v}_2.\]

Then \(\vec{n} = \vec{v}_1 \times \vec{v}_2\) is a normal vector and again we can use **Exp. 1**.]

**Exp. 5** [Given: A parallel plane: \(ax + by + cz = d\)

A point: \((x_0, y_0, z_0)\)

Two parallel planes share normal vectors. So \(<a, b, c>\) is a normal vector of the considered plane. Hence an equation of this plane is

\[ax + by + cz = ax_0 + by_0 + cz_0.\]

**Exp. 6** [Given: Contains two parallel lines]
\( \overrightarrow{r}_1(t) = \overrightarrow{OP}_1 + t \overrightarrow{V} \)

and \( \overrightarrow{r}_2(t) = \overrightarrow{OP}_2 + t \overrightarrow{V} \).

Then \( \overrightarrow{r} = \overrightarrow{P}_1 \overrightarrow{P}_2 \times \overrightarrow{V} \) is a normal vector. Now we can use \( \overrightarrow{r} \) for \( \overrightarrow{P}_1 \) and the point \( \overrightarrow{P}_1 \).

**Vector-valued functions**

A function \( \overrightarrow{r}(t) = \langle x(t), y(t), z(t) \rangle \) is called a vector-valued function (for now).

This type of function is helpful to

- Parametrize a curve
- Describe the position vector of a moving particle.
- Find velocity or acceleration of a moving particle.
- etc.

**Exp 1**. Parametrize the segment \( PQ \) where

\( P = (1,0,1) \) and \( Q = (2,3,4) \).

**Solution**. 

\( \overrightarrow{P}(t) = t \overrightarrow{OQ} + (1-t) \overrightarrow{OP} \) for \( 0 \leq t \leq 1 \).

\( \overrightarrow{r}(t) = t <2,3,4> + (1-t) <1,0,1> \)

\( = <1+t, 3t, 1+3t> \) for \( 0 \leq t \leq 1 \).
Exp 2. Parametrize the line through \( P_0 = (1, 2, 3) \) and parallel to \( \vec{v} = \langle 4, 5, 6 \rangle \).

Solution. \( \vec{r}(t) = \overrightarrow{OP_0} + t \vec{v} \)

\[
= \langle 1, 2, 3 \rangle + t \langle 4, 5, 6 \rangle \\
= \langle 1+4t, 2+5t, 3+6t \rangle.
\]

Exp 3. Parametrize the circle of radius 2, centered at the origin in the \( yz \)-plane.

Solution. \( x = 0 \) (Since in the \( yz \)-plane)

\[
y = 2 \cos \theta \quad \text{and} \quad z = 2 \sin \theta \quad \text{for} \quad 0 \leq \theta < 2\pi.
\]

So \( \vec{r}(\theta) = \langle 0, 2 \cos \theta, 2 \sin \theta \rangle \) for \( 0 \leq \theta < 2\pi \).

Remark. Whenever you see \( A^2 + B^2 = 1 \), you should think of \( A = \cos \theta \) and \( B = \sin \theta \)!

Exp 4. Parametrize an ellipse with center \((1, -2, -1)\) which is a translation of the ellipse

\[
\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1
\]

in the \( xy \)-plane.
Solution. A parametrization of the ellipse
\[(\frac{x}{2})^2 + (\frac{y}{3})^2 = 1\]
in the \(xy\)-plane is
\[
\begin{align*}
\frac{x}{2} &= \cos \theta \quad \Rightarrow \quad (2 \cos \theta, 3 \sin \theta, 0) \\
\frac{y}{3} &= \sin \theta \\
z &= 0
\end{align*}
\]
So a parametrization of its translate with center
\[(1, -2, -1)\]
is \(\vec{r}(\theta) = (1, -2, -1) + (2 \cos \theta, 3 \sin \theta, 0)\)
\[= (1 + 2 \cos \theta, -2 + 3 \sin \theta, -1)\]
for \(0 \leq \theta < 2\pi\).

Exp 5. Parametrize the curve of intersection of the
cylinder \(x^2 + y^2 = 1\) and the plane \(x + y + z = 1\).
Solution. \(x = \cos \theta\), \(y = \sin \theta\), for \(0 \leq \theta < 2\pi\).
\(z = 1 - x - y = 1 - \cos \theta - \sin \theta\).
So \(\vec{r}(\theta) = (\cos \theta, \sin \theta, 1 - \cos \theta - \sin \theta)\) for \(0 \leq \theta < 2\pi\).
Exp 6. Parametrize the curve of intersection of the cylinders \( x^2 + y^2 = 1 \) and \( y^2 + z^2 = 1 \), and \( z \geq 0 \).

Solution. \( x = \cos \theta, \ y = \sin \theta \), for \( 0 \leq \theta < 2\pi \).

\[
\begin{align*}
z^2 &= 1 - y^2 = 1 - \sin^2 \theta = \cos^2 \theta \\
\Rightarrow z &= |\cos \theta|.
\end{align*}
\]

Since \( z \geq 0 \), \( z = |\cos \theta| \). So

\[
\vec{r}(\theta) = \left< \cos \theta, \sin \theta, |\cos \theta| \right>.
\]

Calculus of vector-valued functions

\[
\lim_{t \to a} \vec{r}(t) = \left< \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right>
\]

Exp. Find \( \lim_{t \to 0} \left< \frac{e^t - 1}{t}, \frac{\sin t}{t}, \cos t \right> \).

Solution. \( \lim_{t \to 0} \left< \frac{e^t - 1}{t}, \frac{\sin t}{t}, \cos t \right> = \)
\[ \langle \lim_{t \to 0} \frac{e^t - 1}{t}, \lim_{t \to 0} \frac{\sin t}{t}, \lim_{t \to 0} \frac{\cos t}{t} \rangle = \langle 1, 1, 1 \rangle. \]

(L'Hopital's rule)

\[ \langle \lim_{t \to 0} \frac{e^t}{1}, \lim_{t \to 0} \frac{\cos t}{1}, \cos(0) \rangle = \langle 1, 1, 1 \rangle. \]

Derivative \( \overrightarrow{r}'(t_0) = \lim_{h \to 0} \frac{\overrightarrow{r}(t_0 + h) - \overrightarrow{r}(t_0)}{h} \)

So \( \overrightarrow{r}'(t_0) \) is parallel to the tangent line of \( \overrightarrow{r}'(t) \) at \( \overrightarrow{r}'(t_0) \).

Also we have \( \overrightarrow{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \).

Exp. [Parametrization of the tangent line of \( \overrightarrow{r}(t) \) at \( \overrightarrow{r}(t_0) \):

\[ \overrightarrow{l}(t) = \overrightarrow{r}(t_0) + t \overrightarrow{r}'(t_0) \]

Find a parametrization of the tangent line of a helix \( \overrightarrow{r}(t) = \langle \cos(2t), t, \sin(2t) \rangle \) at
\[ \vec{r}(\pi/4) = \langle 0, \pi/4, 1 \rangle. \]

**Solution.**

\[ \vec{L}(t) = \vec{r}(\pi/4) + t \vec{r}'(\pi/4). \]

\[ \vec{r}'(t) = \langle -2 \sin(2t), 1, 2 \cos(2t) \rangle. \]

So

\[ \vec{L}(t) = \langle 0, \pi/4, 1 \rangle + t \langle -2, 1, 0 \rangle = \langle -2t, t + \pi/4, 1 \rangle. \]

**A moving particle**

If \( \vec{r}(t) \) is the position vector of a moving particle, then

- its velocity \( \vec{v}(t) = \vec{r}'(t). \)
- its acceleration \( \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t). \)
- its speed \( \mathbf{s}(t) = \| \vec{v}(t) \| = \| \vec{r}'(t) \|. \)

**Exp.** The position vector of a moving particle is

\[ \vec{r}(t) = \langle \ln t, t+1, t^2 \rangle \]

(a) Find its velocity at \( t=1 \).

(b) Find its acceleration at \( t=1 \).

(c) Find its speed at \( t=1 \).
Solution. \[ r'(t) = \left< \frac{1}{t}, 1, 2t \right> \]
\[ r''(t) = \left< \frac{-1}{t^2}, 0, 2 \right> \]
(a) \[ r'(1) = \left< 1, 1, 2 \right> \]
(b) \[ r''(1) = \left< -1, 0, 2 \right> \]
(c) \[ ||r'(1)|| = \sqrt{1+1+4} = \sqrt{6}. \]

**Warning.** Non-zero acceleration does NOT mean that the speed is not constant. It only means that the velocity is not constant.

Exp. The position vector of a moving particle is \[ \mathbf{r}(t) = \left< \cos(t), \sin(t), t \right> \]

(a) Find its speed.

(b) Find its acceleration.

Solution. (a) \[ s(t) = \| r'(t) \| = \| \left< -\sin(t), \cos(t), 1 \right> \| \]
\[ = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}. \]

(b) \[ a(t) = r''(t) = \left< -\cos t, -\sin t, 0 \right> \]
\[ r'(t) = \left< -\sin t, \cos t, 1 \right> \]
Notice  In the above example, speed is constant, but acceleration is NOT zero.

Q How can we determine if a moving particle is speeding up or slowing down at $t=t_0$?

**Intuition**
- If $\text{Proj}_\vec{v} \vec{a}$ and $\vec{v}$ have the same direction, then it is speeding up.
- If $\text{Proj}_\vec{v} \vec{a}$ and $\vec{v}$ have opposite directions, then it is slowing down.
- If $\text{Proj}_\vec{v} \vec{a} = 0$, then the speed is constant.

**Geometrically**
- $\theta$ acute $\Rightarrow$ speeding up
- $\theta$ obtuse $\Rightarrow$ slowing down
- $\theta = \pi/2$ $\Rightarrow$ constant speed.

**Computational** $\vec{a} \cdot \vec{v} > 0 \Rightarrow$ speeding up
\[ \vec{a} \cdot \vec{v} < 0 \Rightarrow \text{Slowing down} \]
\[ \vec{a} \cdot \vec{v} = 0 \Rightarrow \text{Constant speed} \]

**Mathematical reasoning**

It is speeding up at \( t = t_0 \) if and only if

\[ s'(t_0) > 0 \] where \( s'(t) = \| \vec{v}(t) \| = \| \vec{r}'(t) \| \)

is the speed function.

\[ s'(t) = \frac{d}{dt} \sqrt{\vec{v}(t) \cdot \vec{v}(t)} \]

(Chain rule)

\[ = \frac{1}{2 \sqrt{\vec{v}(t) \cdot \vec{v}(t)}} \frac{d}{dt} (\vec{v}(t) \cdot \vec{v}(t)) \]

(Dot product rule)

\[ = \frac{1}{2 \| \vec{v}(t) \|} (\vec{v}'(t) \cdot \vec{v}(t) + \vec{v}(t) \cdot \vec{v}'(t)) \]

\[ = \frac{2 \vec{a}(t) \cdot \vec{v}(t)}{2 \| \vec{v}(t) \| \| \vec{v}(t) \|} = \frac{\vec{a}(t) \cdot \vec{v}(t)}{\| \vec{v}(t) \|} \]

(This is actually the component of \( \vec{a} \) along \( \vec{v} \).)

The total distance traveled over \( a \leq t \leq b \) is

\[ \int_{a}^{b} \| \vec{r}'(t) \| dt. \]