Summary of Week 5's lectures

The concept of limit for two or more variable functions is similar to the single variable case. Namely we say

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \]

if someone demands for an \( \varepsilon \)-approximation of \( L \) via values of \( f \), you would be able to say that it is enough to get a \( \delta \)-approximation of \( (a,b) \).

Mathematically we write

For any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that

\[ \| (x,y) - (a,b) \| < \delta \Rightarrow |f(x,y) - L| < \varepsilon . \]

In this course, you are expected to be able

1. Compute very easy limits:

   When \( f \) is the composite of well-known
nice functions and no surprises happens when you evaluate \( f(a,b) \).

2) Compute limits of functions of the form
\[
f(x, y) = g(x) \cdot h(y).
\]

3) Use the following to possibly get a simpler limit:
\[
\lim_{(x,y) \to (a,b)} g(f(x,y)) = L \quad \text{if}
\]
\[
\lim_{t \to l} g(t) = L \quad \text{and} \quad \lim_{(x,y) \to (a,b)} f(x,y) = L.
\]

\[
\text{Exp.} \quad \lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1
\]

Since \( \lim_{(x,y) \to (0,0)} x^2 + y^2 = 0 \) and
\[
\lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\cos t}{1} = 1.
\]

4) Use limits along lines to show
\[
\lim_{(x,y) \to (a,b)} f(x,y)
\]
does NOT exist.
If we compute \( \lim_{x \to 0} f(x, cx) \)

and the answer depends on \( c \), then

\[ \lim_{(x,y) \to (a,0)} f(x,y) \]

does not exist.

**Remark.** For a limit when \((x,y) \to (a,b) \neq (c,0)\)
you have to use the lines

\[ y = c(x-a) + b. \]

So you have to compute

\[ \lim_{x \to a} f(x, c(x-a)+b) \]

and see if it depends on \( c \) or not.

5) Sometimes limit along all the lines are
the same. It might be helpful to try
other curves. For instance, if \( f(x,y) \)
is ratio of two polynomials, for

\[ \lim_{(x,y) \to (0,0)} f(x,y) \]
it is helpful to consider
\[ y = x^c \text{ or } x = y^c \text{ so that you get the same deg in the numerator and the denom.} \]

**Example.** Show that 
\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3 y}{x^6 + y^2}
\]
does NOT exist.

**Solution.** Along \( y = cx \)
\[
\lim_{{x \to 0}} \frac{x^3 (cx)}{x^6 + c^2 x^2} = \lim_{{x \to 0}} \frac{c x^2}{x^4 + c^2} = 0.
\]

Along \( y = x^3 \)
\[
\lim_{{x \to 0}} \frac{(x^3)(x^3)}{x^6 + x^6} = \frac{1}{2}
\]

Since \( 0 \neq \frac{1}{2} \), the two-variable limit does NOT exist.

6) Use polar coordinates and squeeze theorem to compute a limit or to show it does NOT exist.

\[
\lim_{{(x,y) \to (0,0)}} f(x,y) = \lim_{{r \to 0}} f(r \cos \theta, r \sin \theta) \text{ uniform on } \theta
\]
First, compute \( f(r \cos \theta, r \sin \theta) \)

Second, find \( q_1(r) \) and \( q_2(r) \) such that

\[
\text{A} \quad \lim_{r \to 0} q_1(r) = \lim_{r \to 0} q_2(r) = L
\]

\[
\text{B} \quad q_1(r) \leq f(r \cos \theta, r \sin \theta) \leq q_2(r)
\]

Third, use Squeeze Theorem and conclude that

\[
\lim_{(x,y) \to (0,0)} f(x,y) = L.
\]

**Example.** Let \( a \geq 0, \ b \geq 0 \). Show that

\[
\lim_{(x,y) \to (0,0)} \frac{x^a y^b}{x^2 + y^2} \begin{cases} \text{does NOT exist} & \text{if } a + b \leq 2 \\ = 0 & \text{if } a + b > 2. \end{cases}
\]

**Solution.**

\[
\lim_{(x,y) \to (0,0)} \frac{x^a y^b}{x^2 + y^2} = \lim_{r \to 0} \frac{a + b}{r^{a+b-2}} \cos^a \theta \cdot \sin^b \theta
\]

\[
= \lim_{r \to 0} r^{a+b-2} \cos^a \theta \cdot \sin^b \theta \text{ uniform on } \theta
\]
If \( a + b < 2 \), even for \( \theta = \frac{\pi}{4} \)

\[
\lim_{r \to 0} \frac{\cos^a(\frac{\pi}{4}) \sin^b(\frac{\pi}{4})}{r^{2-(a+b)}} = \infty.
\]

If \( a + b = 2 \), then for a fixed \( \theta \) we have

\[
\lim_{r \to 0} \cos^a \theta \sin^b \theta = \cos^a \theta \sin^b \theta
\]

which depends on \( \theta \). So

\[
\lim_{(x,y) \to (0,0)} \frac{x^a y^{2-a}}{x^2 + y^2}
\]

does not exist.

If \( a + b > 2 \), then

\[
0 \leq |r^{a+b-2} \cos^a \theta \sin^b \theta| \leq r^{a+b-2}
\]

and

\[
\lim_{r \to 0} 0 = \lim_{r \to 0} r^{a+b-2} = 0.
\]

Hence

by Squeeze theorem,

\[
\lim_{(x,y) \to (0,0)} \frac{x^a y^b}{x^2 + y^2} = 0.
\]
\[
\text{Exp. } \lim_{(x,y) \to (0,0)} 8\sin(xy) \cos\left(\frac{1}{x^2+y^2}\right)
\]

Solution. \[0 \leq |f(x,y)| \leq 8\sin(xy) \to 0\]

By the squeeze theorem, we have \[\lim_{(x,y) \to (0,0)} f(x,y) = 0. \]

Partial derivatives

In order to understand the behavior of \(f(x,y)\) around the point \((a,b)\), first we focus on one variable at a time. We fix \(y = b\) and let \(x\) vary to find the rate of change of \(f\) with respect to \(x\), i.e., find the derivative of \(f(x,b)\).

Geometrically we look at the curve of intersection of \(z = f(x,y)\) and \(y = b\), and parametrize it using \(x\) as a parameter:

\[\vec{r}(x) = \langle x, b, f(x,b) \rangle\]

Now we would like to find the vector parametrization of its tangent
line at $x=a$: $\vec{r}(a)+t\vec{r}'(a)$

$\vec{r}'(x) = <1,0,0>$ is the partial derivative of $f$ with respect to $x$. It is denoted by $\frac{\partial f}{\partial x}$ or $f_x$. So the tangent line is $<a,b,f(a,b)>+t<1,0,f_x(a,b)>$.

A vector parametrization of the curve of intersection of

\[
\begin{align*}
z & = f(x,y) \\
x & = a
\end{align*}
\]

is $<a,y,f(a,y)>$. So its tangent line at $y=b$ is $<0,1,f_y(a,b)>$.

**Partial derivative from computational point of view**

\[
\begin{align*}
\text{Exp. } z &= x^2+xy-y^2 \Rightarrow f_x &= 2x+y \\
f_y &= x-2y
\end{align*}
\]

\[
\begin{align*}
\text{Exp. } z &= \sin\left(\frac{x}{y}\right) \Rightarrow f_x &= \frac{1}{y} \cos\left(\frac{x}{y}\right) \quad [\text{Chain rule}] \\
f_y &= -\frac{x}{y^2} \cos\left(\frac{x}{y}\right)
\end{align*}
\]
Higher order partial derivatives

$f_x$ is a two (or more) variable function. So we can ask about its partial derivatives:

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$

Notice: \( \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \)

Example. \( f(x,y) = \ln(x^2 + y^2) \). Find all the second partial derivatives.

Solution. \( f_x = \frac{2x}{x^2 + y^2} \) (Chain rule)

\( f_y = \frac{2y}{x^2 + y^2} \)

\( f_{xx} = (f_x)_x = \frac{(2)(x^2 + y^2) - (2x)(2x)}{(x^2 + y^2)^2} = 2 \frac{y^2 - x^2}{(x^2 + y^2)^2} \)
\[ f_{xy} = (f_x)_y = \frac{(0)(x^2+y^2) - (2x)(2y)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2} \]

\[ f_{yx} = (f_y)_x = \frac{(0)(x^2+y^2) - (2y)(2x)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2} \]

\[ f_{yy} = (f_y)_y = \frac{(2)(x^2+y^2) - (2y)(2x)}{(x^2+y^2)^2} = 2 \frac{x^2-y^2}{(x^2+y^2)^2} \]

We observe that the second partial derivatives of the above function satisfy:

1. \( f_{xy} = f_{yx} \)
2. \( f_{xx} + f_{yy} = 0 \)

The first property is fairly general. The second property, however, holds only for a special class of functions called **Harmonic Functions**.

**Theorem** If \( f_{xy} \) and \( f_{yx} \) are continuous in a disk around \((a, b)\), then \( f_{xy}(a, b) = f_{yx}(a, b) \).

**Exp.** [Changing the order might help]

Find \( f_{xy} \) where \( f(x,y) = e^{(\ln x)^2} + xy \)

**Solution**. It is clear that \( f_{xy} \) and \( f_{yx} \) are
continuous in the domain of $f$. So

$$f_{xx} = f_{yx} = (f_y)_x = \frac{1}{f_y}$$

\[ f_y = x \]

**Tangent plane.**

Since graph $z = f(x, y)$ of $f$ is a surface, it is more interesting to find a tangent plane rather than tangent lines.

If there is a tangent plane then it should contain the tangent lines that we found above:

$$<a, b, f(a, b)> + t <1, 0, f_x(a, b)> \quad \text{and} \quad <a, b, f(a, b)> + t <0, 1, f_y(a, b)>.$$ 

To find the equation of this plane we need to compute its normal vector:

$$\vec{n} = <1, 0, f_x(a, b)> \times <0, 1, f_y(a, b)>$$

$$= \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 0 & f_x(a, b) \\
0 & 1 & f_y(a, b)
\end{vmatrix} = <-f_x(a, b), -f_y(a, b), 1>.$$
So the equation of the possible tangent plane is
\[ \vec{n} \cdot \langle x, y, z \rangle = \vec{n} \cdot \langle a, b, f(a,b) \rangle \]
\[ \text{a point in the tangent plane} \]
\[ \Rightarrow \vec{n} \cdot \langle x-a, y-b, z-f(a,b) \rangle = 0 \]
\[ \Rightarrow -f_x(a,b)(x-a) - f_y(a,b)(y-b) + z - f(a,b) = 0 \]
\[ \Rightarrow z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]
So the tangent plane at the point \((a, b, f(a,b))\) is the graph of the function
\[ L(x, y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]
Hence \( L(x, y) \) is a good approximation of \( f(x,y) \) if \( f \) has a tangent plane.