Math 10B. Lecture Examples.

Section 11.5. Growth and Decay[†]

Example 1 Match problems **(I)** through **(IV)** below to differential equations (a) through (d) and to the slope fields in Figures 1 through 4.

(I) The thickness of the ice on a lake grows at a rate that is proportional to the reciprocal of its thickness. Find the thickness $y = y(t)$ as a function of the time t.

(II) A population grows at a rate proportional to its size. Find the population $y = y(t)$ as a function of the time t.

(III) A hot potato is taken out of the oven at time $t = 0$ into a kitchen that is at 20° Celsius. The rate of change of the potato's temperature is proportional to the difference between its temperature and that of the kitchen. Find the temperature $y = y(t)$ of the potato as a function of t.

(IV) Find a function $y = y(t)$ whose rate of change with respect to t is $-2t$.

(a) $\dfrac{dy}{dt} = 0.2y$ **(b)** $\dfrac{dy}{dt} = \dfrac{20}{y}$

(c) $\dfrac{dy}{dt} = -2t$ **(d)** $\dfrac{dy}{dt} = -2(y - 20)$

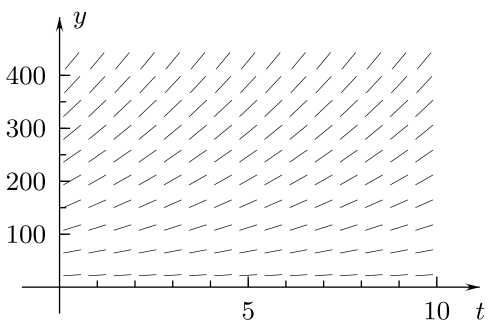


FIGURE 1

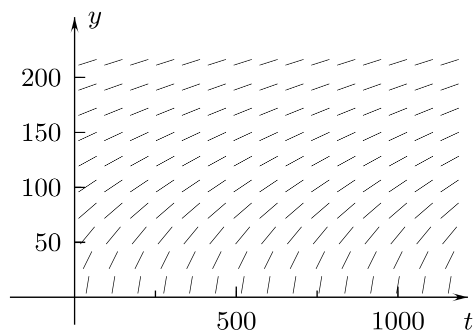


FIGURE 2

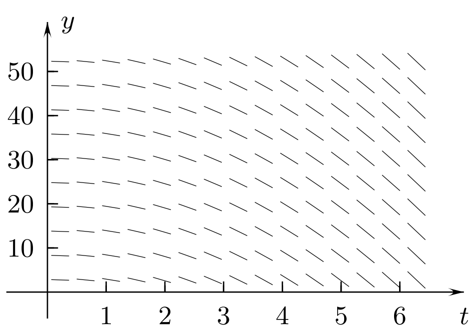


FIGURE 3

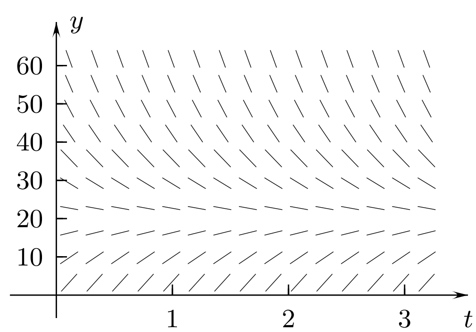


FIGURE 4

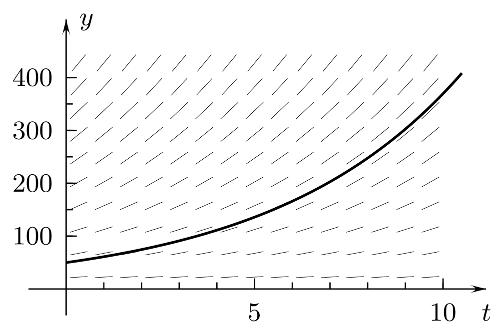
Answer: Problem I goes with equation (b) and the slope field in Figure 2. • Problem II goes with equation (a) and Figure 1. • Problem III goes with equation (d) and Figure 4. • Problem IV goes with equation (c) and Figure 3.

[†]Lecture notes to accompany Section 11.5 of *Calculus* by Hughes-Hallett et al.

Example 2 **(a)** Solve the differential equation **(a)** $\frac{dy}{dt} = 0.2y$ from Example 1 (a population) with the initial condition $y(0) = 50$ and draw its graph with the corresponding slope field. **(b)** What happens to the solution as $t \to \infty$?

Answer: (a) $y = 50e^{0.2t}$ • Figure A2. **(b)** The population $y = 50e^{0.2t}$ tends to ∞ as $t \to \infty$.

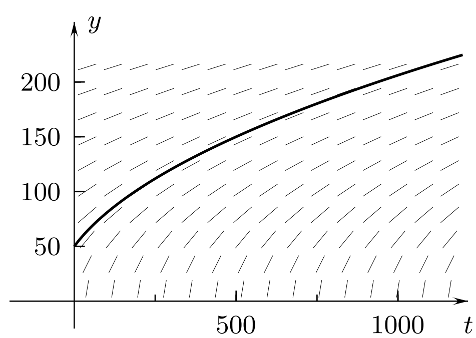
Figure A2



Example 3 Solve differential equation **(b)** $\frac{dy}{dt} = \frac{20}{y}$ from Example 1 (ice thickness) with the initial condition $y(0) = 50$ and draw its graph with the corresponding slope field.

Answer: $y = \sqrt{40t + 2500}$ • Figure A3

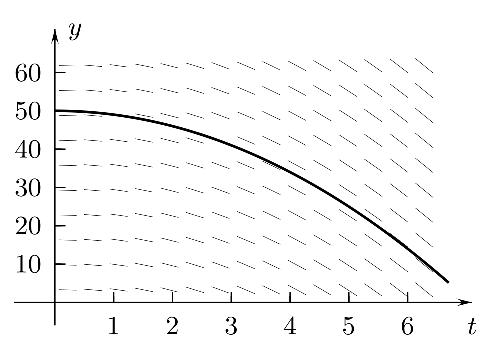
Figure A3



Example 4 Solve differential equation **(c)** $\frac{dy}{dt} = -2t$ from Example 1 (a function) with the initial condition $y(0) = 50$ and draw its graph with the corresponding slope field.

Answer: $y = 50 - t^2$ • Figure A4

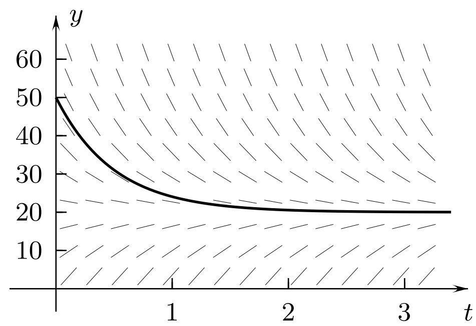
Figure A4



Example 5 **(a)** Solve differential equation **(d)** $\frac{dy}{dt} = -2(y-20)$ from Example 1 (the potato) with the initial condition $y(0) = 50$ and draw the graph of its solution with the corresponding slope field. **(b)** What happens to the solution as $t \to \infty$ and why is this plausible?

Answer: (a) $y = 20 + 30e^{-2t}$ • Figure A5.
(b) y tends to 20 as $t \to \infty$. (This is plausible because y is the potato's temperature, which tends to the room temperature 20°C as $t \to \infty$.)

Figure A5



Example 6 Match problems **(I)**, **(II)**, and **(III)** with differential equations **(a)**, **(b)**, and **(c)** and the slope fields in Figures 5 through 7.

(I) v is the downward velocity, measured in meters per second, of a ball that is falling under the force of gravity but with no air resistance or other forces on it. The time t is measured in seconds and the acceleration due to gravity is 9.8 meters per second^2.

(II) v is the downward velocity, measured in meters per second, of a suitcase that is falling under the force of gravity with air resistance that is proportional to its velocity. The suitcase's downward acceleration is zero when its downward velocity is 49 meters per second. (This is called the EQUILIBRIUM VELOCITY because it is the velocity at which the upward force of air resistance equals the downward force of gravity. It is also referred to as the TERMINAL VELOCITY because it is the limit of the suitcase's velocity as $t \to \infty$.)

(III) v is the downward velocity, measured in meters per second, of a rock that is falling under the force of gravity with air resistance that is proportional to its velocity. Its equilibrium velocity is 98 meters per second.

$$\textbf{(a)}\quad \frac{dv}{dt} = 9.8 \qquad \textbf{(b)}\quad \frac{dv}{dt} = 9.8 - \tfrac{1}{10}v \qquad \textbf{(c)}\quad \frac{dv}{dt} = 9.8 - \tfrac{1}{5}v.$$

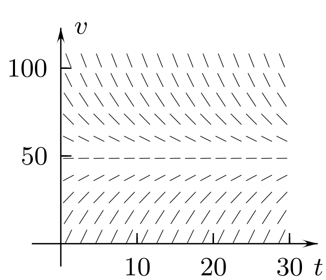


FIGURE 5

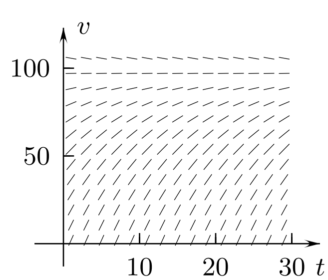


FIGURE 6

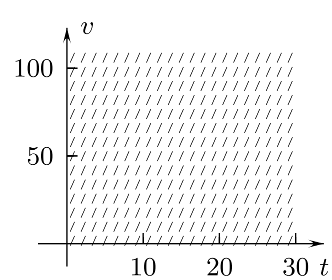


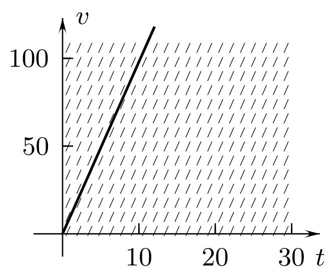
FIGURE 7

Answer: Problem I goes with differential equation (a) and Figure 7. • Problem II goes with equation (c) and Figure 5. • Problem III goes with equation (b) and Figure 6.

Example 7 Find the solution of differential equation **(a)** $\dfrac{dv}{dt} = 9.8$ in Example 6 with the initial condition $v(0) = 0$. Draw its graph with the corresponding slope field.

Answer: $v = 9.8t$ • Figure A7

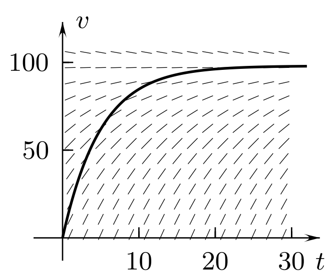
$v = 9.8t$
Figure A7



Example 8 Solve the initial-value problem, $\dfrac{dv}{dt} = 9.8 - \frac{1}{10}v, v(0) = 0$ for differential equation (b) in Example 6. Then draw the graph of the solution with the corresponding slope field.

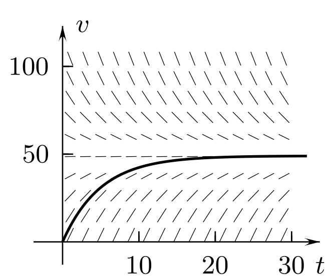
Answer: $v = 98 - 98e^{-t/10}$ (which tends to 98 as $t \to \infty$) • Figure A8

$v = 98 - 98e^{-t/10}$
Figure A8



Example 9 Solve the initial-value problem, $\dfrac{dv}{dt} = 9.8 - \frac{1}{5}v, v(0) = 0$ for differential equation (c) in Example 6 and draw the graph of the solution with the corresponding slope field.

Answer: $v = 49 - 49e^{-t/5}$ (which tends to 49 as $t \to \infty$) • Figure A9



$v = 49 - 49e^{-t/5}$
Figure A9

Example 10 Match problems **(I)** and **(II)** below with differential equations **(a)** and **(b)** and the slope fields in Figures 8 and 9.

(I) v is the horizontal velocity, measured in miles per hour, of a model car whose acceleration, due to its faltering engine, oscillates between 0 and 90 miles per hour^2. There are no other forces on the car and time is measured in hours.

(II) v is the horizontal velocity, measured in feet per minute, of a motor boat that has its engine turned off and is slowing down because of water and air resistance. The resistance is proportional to the square of the boat's velocity, and there are no other forces on it. Time is measured in minutes.

$$\textbf{(a)}\quad \frac{dv}{dt} = -\tfrac{1}{8}v^2, \qquad \textbf{(b)}\quad \frac{dv}{dt} = 45 + 45\cos(15t)$$

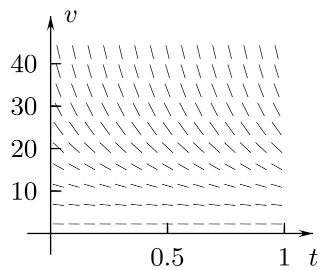


FIGURE 8

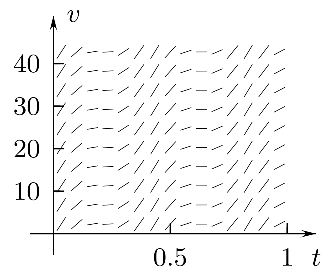
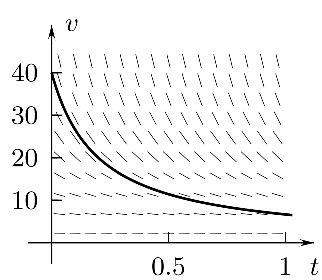


FIGURE 9

Answer: Problem I goes with differential equation (b) and Figure 9. • Problem II goes with equation (a) and Figure 8.

Example 11 Find the solution of equation **(a)** $\dfrac{dv}{dt} = -\frac{1}{8}v^2$ from Example 10 with the initial condition $v(0) = 40$. Draw its graph with the corresponding slope field.

Answer: $v = \dfrac{40}{5t+1}$ • Figure A11 (Notice that $v \to 0$ as $t \to \infty$.)

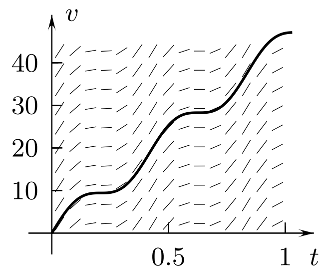


$v = \dfrac{40}{5t+1}$

Figure A11

Example 12 Solve equation **(b)** $\frac{dv}{dt} = 45 + 45\cos(15t)$ from Example 10 with the initial condition $v(0) = 0$ and draw the graph of the solution with the corresponding slope field.

Answer: $v = 45t + 3\sin(15t)$ • Figure A12



$v = 45t + 3\sin(15t)$

Figure A12