

Math 10C. Lecture Examples.

Sections 12.1, 12.2, and 12.3. Functions of two variables[†]

Example 1 (a) What is the domain of $f(x, y) = x^2 + y^2$? (b) What are the values $f(2, 3)$ and $f(-2, -3)$ of this function at $(2, 3)$ and $(-2, -3)$? (c) What is its range?

Answer: (a) The domain of f is the entire xy -plane. (b) $f(2, 3) = 13$ • $f(-2, -3) = 13$. (c) The range of f is the closed infinite interval $[0, \infty)$.

Example 2 Determine the shape of the surface $z = x^2 + y^2$ in xyz -space by studying its cross sections in the planes $x = c$ perpendicular to the x -axis.

Answer: The intersection of the surface $z = x^2 + y^2$ with the plane $x = c$ is a parabola that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy -plane if $c \neq 0$ • Figure A2a • The surface has the bowl-like shape in Figure A2b

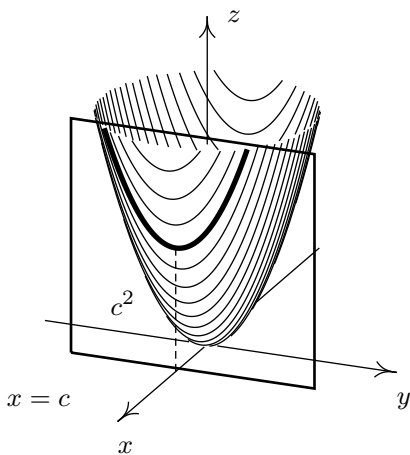


Figure A2a

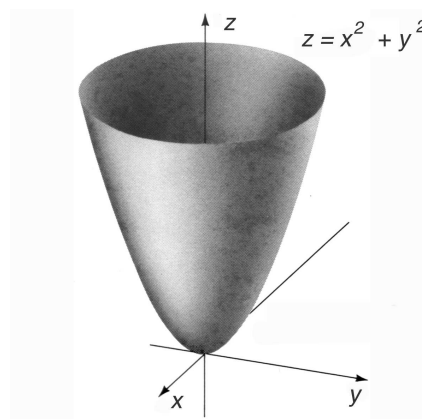


Figure A2b

[†]Lecture notes to accompany Sections 12.1, 12.2, and 12.3 of *Calculus* by Hughes-Hallett et al.

Example 3 Determine the shape of the surface $z = x^2 + y^2$ of Example 2 by studying its cross sections in the planes $y = c$ perpendicular to the y -axis.

Answer: The intersection of the surface $z = x^2 + y^2$ with the plane $y = c$ is parabola that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy -plane if $c \neq 0$. • Figure A3a • The surface has the bowl-like shape from Example 2. (Figure A3b shows the cross sections from Examples 2 and 3 together.)

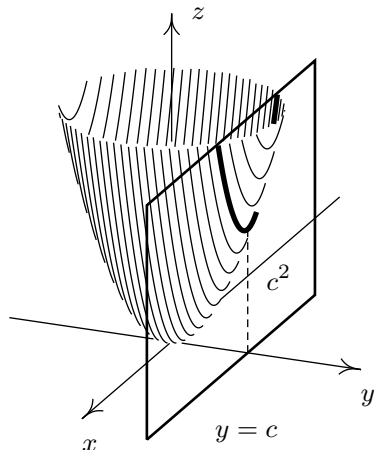


Figure A3a

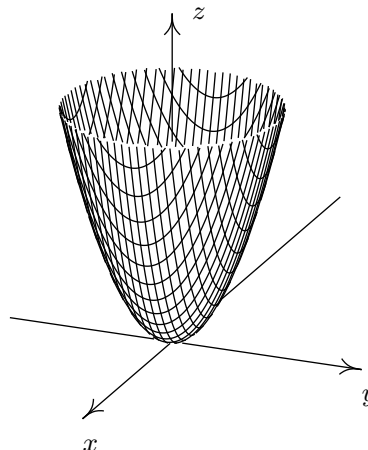


Figure A3b

Example 4 Determine the shape of the surface $z = y^2 - x^2$ by studying its cross sections in the planes $x = c$ perpendicular to the x -axis.

Answer: The intersection of the surface $z = y^2 - x^2$ with the plane $x = c$ is a parabola that opens upward and whose vertex is c^2 units below the xz -plane. • Figure A4a • The vertex is at the origin for $c = 0$ and drops below the xy -plane as c moves away from zero. • The surface has the saddle shape in Figure A4b.

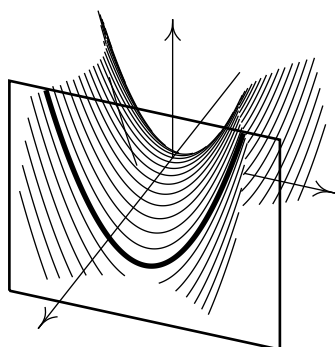


Figure A4a

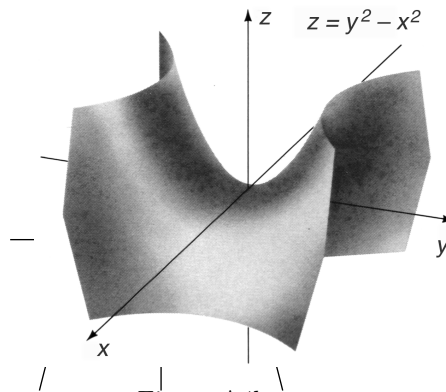


Figure A4b

Example 5 Determine the shape of the surface $z = y^2 - x^2$ of Example 4 by studying its cross sections in the planes $y = c$ perpendicular to the y -axis.

Answer: The intersection of the surface $z = y^2 - x^2$ with the plane $y = c$ is a parabola that opens downward and whose vertex is c^2 units above the xy -plane. • Figure A5a • The vertex is at the origin for $c = 0$ and rises above the xy -plane as c moves away from zero. • The surface has the saddle shape from Example 4. (Figure A5b shows the two sets of cross sections together.)

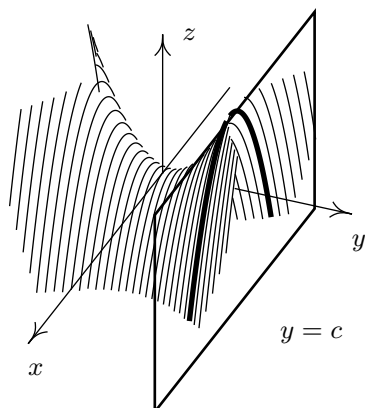


Figure A5a

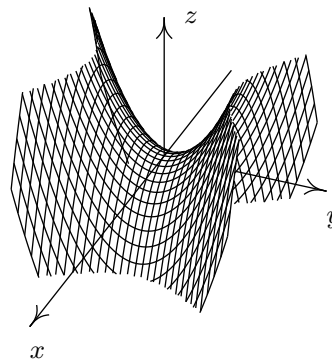


Figure A5b

Example 6 Use the curve $z = y - \frac{1}{12}y^3$ in the yz -plane of Figure 1 to determine the shape of the surface $z = y - \frac{1}{12}y^3 - \frac{1}{4}x^2$.

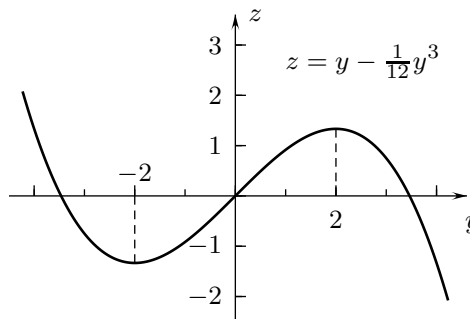


FIGURE 1

Answer: One solution: The cross section of the surface in the plane $x = c$ has the shape of the curve in Figure 1 if $c = 0$, is that curve moved down and forward if $c > 0$ and is that curve moved down and back if $c < 0$. • The surface has the boot-like shape in Figure A6
 Another solution: The cross section in the plane $y = c$ is a parabola that opens downward and has its vertex on the curve in Figure 1. • The surface has the boot-like shape in Figure A6.

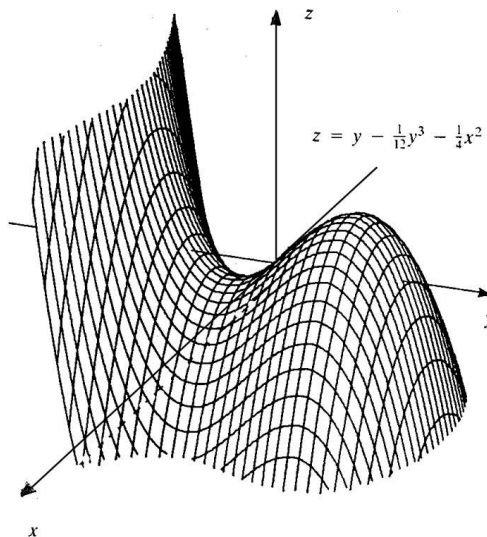


Figure A6

Example 7 Describe the level curves of the function $f(x, y) = x^2 + y^2$ from Examples 2 and 3.

Answer: Figure A7a shows horizontal cross sections of the graph of f and Figure A7b shows the corresponding level curves. • The level curve $f = c$ is the circle of radius \sqrt{c} with its center at the origin if $c > 0$, is the origin if $c = 0$, and is empty if $c < 0$. (The surface is called a “circular paraboloid.”)

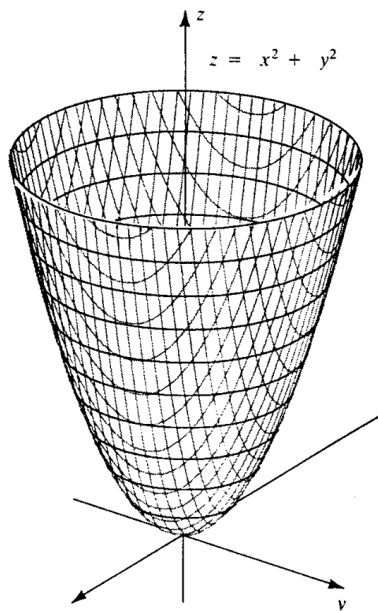


Figure A7a

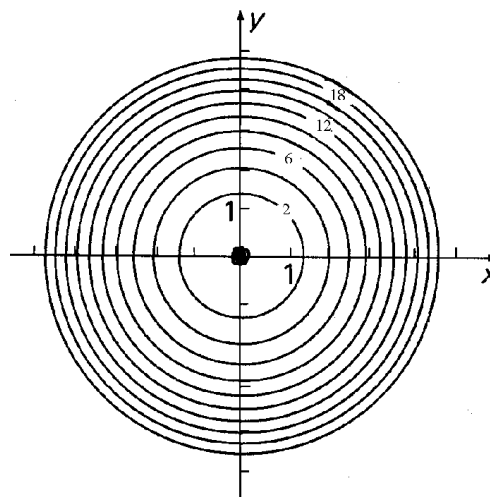


Figure A7b

Example 8 Describe the level curves of $g(x, y) = y^2 - x^2$ from Examples 4 and 5.

Answer: Figures A8a and A8b • The level curve $g = c$ is a hyperbola with the equation $y^2 - x^2 = c$. (The surface is a “hyperbolic paraboloid.”)

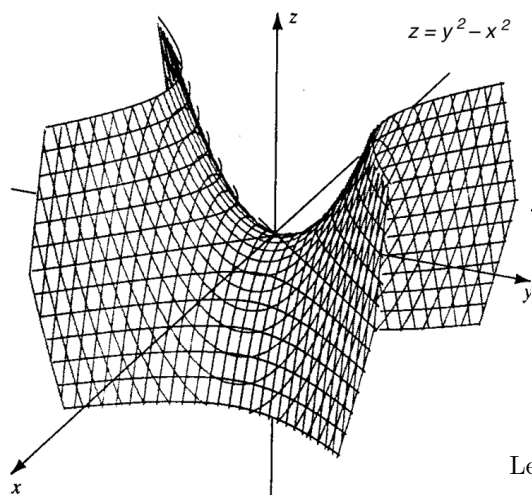
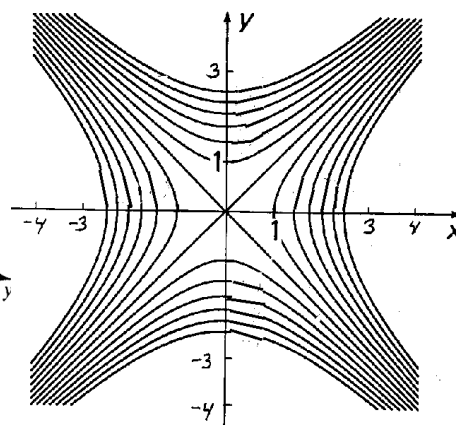


Figure A8a



Level curves of $g(x, y) = y^2 - x^2$

Figure A8b

Example 9 Figures 2 and 3 show horizontal cross sections of the graph of $h(x, y) = y - \frac{1}{12}y^3 - \frac{1}{4}x^2$ from Example 6 and the corresponding level curves. Describe how the surface can be reconstructed from the level curves.

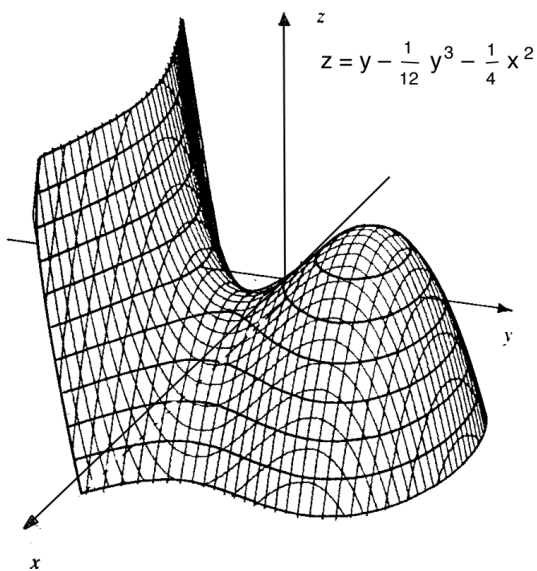


FIGURE 2

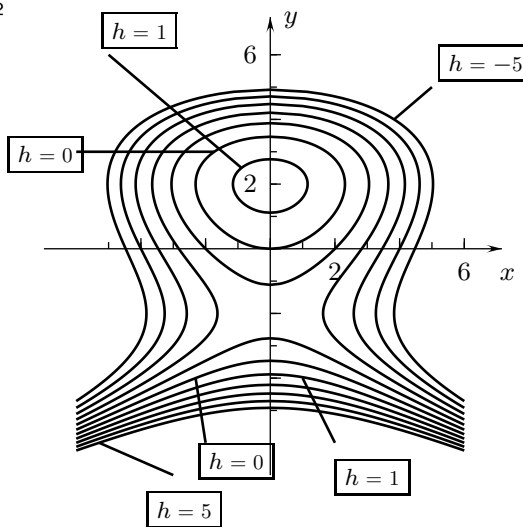


FIGURE 3

Answer: Leave the two parts of the level curve $h = 0$ on the xy -plane. • Raise the two parts of the curve labeled $h = 1$ one unit on the “toe” and “leg” of the “boot.” • Lower the curves above the upper part of $h = 1$ to form the sides of the “boot.” • Raise the curves below the lower part of $h = 1$ to form the more of the “leg” of the “boot.”

Interactive Examples

Work the following Interactive Examples on Shenk’s web page, <http://www.math.ucsd.edu/~ashenk/>:[‡]

Section 14.1: Examples 1–6

[‡]The chapter and section numbers on Shenk’s web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.