(11/25/08)

Math 10C. Lecture Examples.

Section 15.3. Lagrange multipliers[†]

Imagine that the curve in Figure 1 is a mirror and that a viewer at point F_2 is looking at the image in the mirror of an object at point F_1. According to FERMAT'S PRINCIPLE from physics, the image will be the point P_0 on the mirror such that the total distance

$$f(P) = \overline{PF_1} + \overline{PF_2} \tag{1}$$

that the light travels from the object to the viewer is a minimum a that point. For each number c that is greater than the distance $\overline{F_1F_2}$ between the object and the viewer, the level curve $\overline{PF_1} + \overline{PF_2} = c$ of the distance $f(P)$ is an ellipse. Figure 2 shows eight of these ellipses.[†]

Example 1 **Why can you expect the ellipse to be tangent to the mirror at the point $\mathbf{P_0}$?**

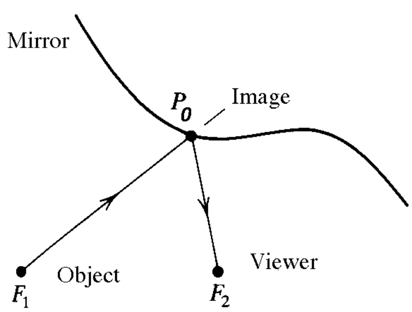

FIGURE 1

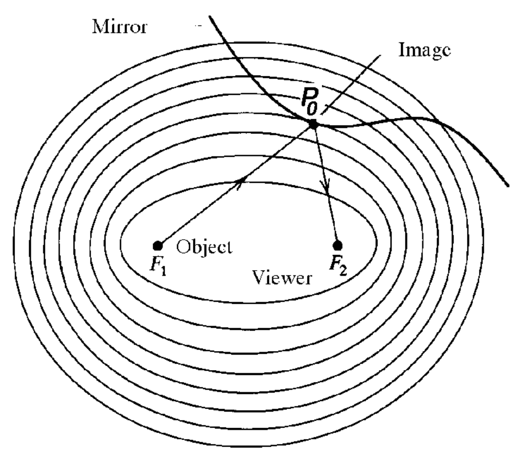

FIGURE 2

Answer: The solution is the answer

To express the result of Example 1 with formulas, we introduce xy-axes, as in Figure 3, and let $f(x,y)$ be the sum **(1)** of the distances from $P = (x,y)$ to F_1 and F_2. We also assume that the mirror C is a level curve $g(x,y) = c$ of another function with a nonzero gradient vector.

At $P_0 = (a,b)$ where the smallest ellipse and C are tangent, $\nabla f(a,b)$ is perpendicular to the ellipse, which is the level curve of f, and $\nabla g(a,b)$ is perpendicular to C, which is the level curve of g (Figure 3). Since the curves are tangent at (a,b), the two gradient vectors are parallel, and there is a number λ such that $\nabla f(a,b) = \lambda\, \nabla g(a,b)$. The number λ is called a LAGRANGE MULTIPLIER. This illustrates the following principle that is used to find maxima and minima of general functions $z = f(x,y)$ on curves.

[†]Lecture notes to accompany Section 15.3 of *Calculus* by Hughes-Hallett et al.

[†]The ellipse $\overline{PF_1} + \overline{PF_2} = c$ could be drawn by fastening the ends of a string of length c at F_1 and F_2 and running a pencil point around inside the taut string.

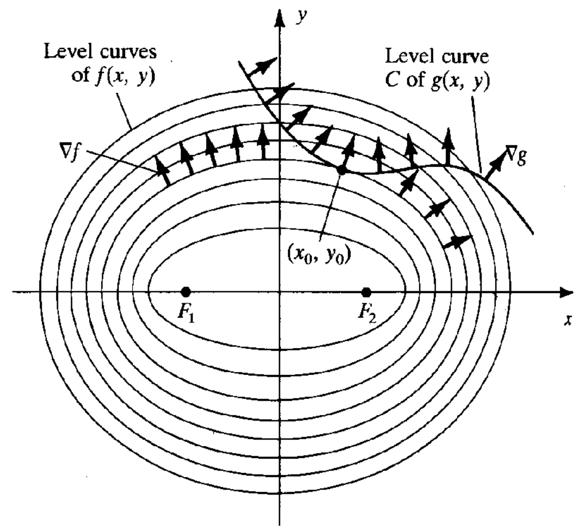

FIGURE 3

Theorem *Suppose that C is a level curve of $z = g(x,y)$, that $\nabla g(x,y)$ is not zero on C, and that $z = g(x,y)$ and $z = f(x,y)$ have continuous first derivatives near C. Then, if $z = f(x,y)$ has a local maximum or a local minimum for (x,y) on C, then that maximum or minimum occurs at a point (a,b) where*

$$\nabla f(a,b) = \lambda \nabla g(a,b) \tag{2}$$

for some number λ.

The curve C in Theorem 1 is called the CONSTRAINT CURVE. The number λ in **(2)** is zero if $\nabla f(a,b)$ is zero and (a,b) is a critical point of f.

Example 2 **Use Lagrange multipliers to find the maximum and minimum values of $\mathbf{f(x,y) = x - 2y + 1}$ on the ellipse $\mathbf{x^2 + 3y^2 = 21}$ and where they occur.**

Answer: The maximum of f on the ellipse is 8 at $(3,-2)$ and the minimum is -6 at $(-3,2)$.

Example 3 **Sketch the ellipse in Example 2 and the level curves of f where the maximum and minimum occur.**

Answer: Figure A3

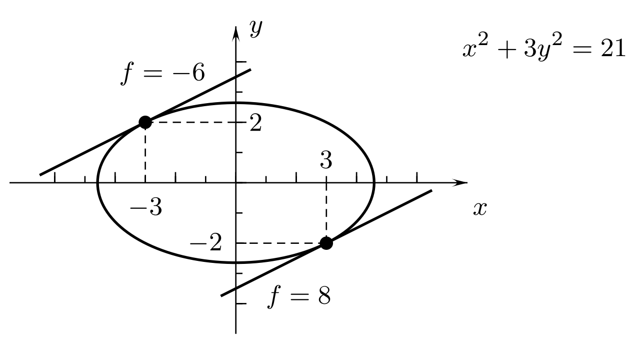

Figure A3

Example 4 **Use Lagrange multipliers to find the rectangle of perimeter 12 that has the smallest area (Figure 4).**

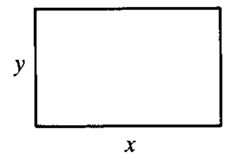

FIGURE 4

Answer: The rectangle with perimeter 12 and maximum area is a square of width 3 and area 9.

Example 4 **Use Lagrange multipliers to find the dimensions of the rectangle of area 9 that has the shortest perimeter.**

Answer: The rectangle of area 9 and minimum perimeter is the square of width 3, and perimeter 12.

In the case of Example 3, the level curves of the area $A = xy$ are the hyperbolas $xy = k$ and the constraint curve is the line $2x + 2y = 12$ in Figure 5. In the case of Example 4, the level curves are the lines $2x + 2y = c$ and the constraint curve is the hyperbola $xy = 9$ in Figure 6.

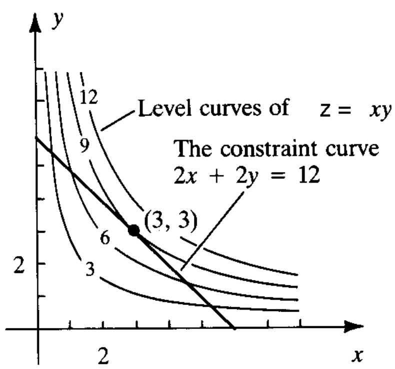

FIGURE 5

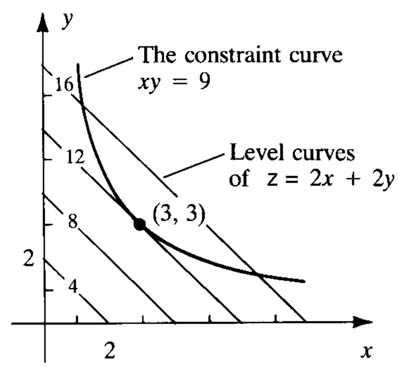

FIGURE 6

Example 4 **Solve Example 2 by finding the minimum of a function of one variable.**

Answer: The constraint condition $2x + 2y = 12$ implies that $y = 6 - x$, which in the formula $A = xy$ for the area gives $A(x) = x(6 - x) = 6x - x^2$. • $y = A(x)$ has a global maximum for $x = 3$, where $y = 6 - x = 3$.

Example 5 **Find the maximum and minimum of $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^2 + 2\mathbf{x} + \mathbf{y}^2$ for $3\mathbf{x}^2 + 2\mathbf{y}^2 = 48$.**

Answer: [Maximum] $= f(2, \pm\sqrt{18}) = 26$ • [Minimum] $= f(-4, 0) = 8$