Exam 1 Solutions

1) (a) For x in the interval $0 < x < 2$, $f(x) = \frac{8}{x^2}$ so that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{8}{x^2}.$$

The function $y = \frac{8}{x^2}$ is continuous at $x = 2$ so

$$\lim_{x \to 2^-} \frac{8}{x^2} = \frac{8}{2^2} = \frac{8}{4} = 2.$$

Therefore,

$$\lim_{x \to 2^-} f(x) = 2.$$

(b) Using the same approach as used in part (a) we find $\lim\limits_{x \to 2^+} f(x) = 2$.

(c) Recall $\lim\limits_{x \to 2} f(x)$ exists if and only if $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$. If $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$ then $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2^-} f(x)$. Since $\lim\limits_{x \to 2^-} f(x) = 2$ by part (a) and $\lim\limits_{x \to 2^+} f(x) = 2$ by part (b), we have

$$\lim_{x \to 2} f(x) = 2.$$

(d) The function $y = \frac{8}{x^2}$ is continuous for all x except when $x = 0$ hence $y = \frac{8}{x^2}$ is continuous on the interval $(0, 2)$. The function $y = x$ is continuous for all x hence $y = x$ is continuous on the interval $[2, 6]$. Since $f(2) = 2$ and $\lim\limits_{x \to 2} f(x) = 2$ by part (c), f is continuous at $x = 2$. The largest interval on which f is continuous is $(0, \infty)$.

2) Let $f(x) = \frac{1}{x^2+1}$. By definition, $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$. Applying this definition with $f(x) = \frac{1}{x^2+1}$ and $a = 0$:

$$f'(0) = \lim_{x \to 0} \frac{\frac{1}{x^2+1} - \frac{1}{0^2+1}}{x - 0} = \lim_{x \to 0} \frac{\frac{1}{x^2+1} - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{x^2+1} - \frac{x^2+1}{x^2+1}}{x} = \lim_{x \to 0} \frac{\frac{-x^2}{x^2+1}}{x} = \lim_{x \to 0} \frac{-x}{x^2 + 1}.$$

The function $y = \frac{-x}{x^2+1}$ is continuous at $x = 0$ so that

$$\lim_{x \to 0} \frac{-x}{x^2 + 1} = \frac{-0}{0^2 + 1} = 0.$$

We conclude $y'(0) = 0$.

3) We are trying to estimate the value $v'(280)$ using a table of values. The quantity $v'(280)$ is the slope of the tangent line to $v(T)$ when $T = 280$. A reasonable estimate given the table is

$$v'(280) \approx (\text{the slope of the secant line through } (240, 5000) \text{ and } (320, 4724)) .$$

The slope of the secant line through $(240, 5000)$ and $(320, 4724)$ is

$$\frac{4724 - 5000}{320 - 240} = \frac{-276}{80} = -\frac{69}{20}.$$

Therefore $v'(280) \approx -\frac{69}{20}$ and the approximate rate of change of velocity with respect to the temperature is $-\frac{69}{20}$ feet per second per degree. Remark: Here we computed a centered difference quotient. One could have also computed a left-difference quotient using the points $(280, 4879)$ and $(240, 5000)$, or a right-difference quotient using the points $(280, 4879)$ and $(320, 4724)$.

4) The equation of the tangent line at $x = 1$ is $y = y(1) + y'(1)(x - 1)$. We need to compute $y(1)$ and $y'(1)$.

$$y(1) = (1 + 1 - 1^4)(1 - 1 + 1^3) = (1)(1) = 1.$$

To compute $y'(7)$ we need to find $y' = \frac{d}{dx}((1 + x - x^4)(1 - x + x^3))$. To compute y' we use the Product Rule.

$$y' = (1-x+x^3)\frac{d}{dx}(1+x-x^4)+(1+x-x^4)\frac{d}{dx}(1-x+x^3) = (1-x+x^3)(1-4x^3)+(1+x-x^4)(-1+3x^2).$$

Therefore

$$y'(1) = (1 - 1 + 1^3)(1 - 4 \cdot 1^3) + (1 + 1 - 1^4)(-1 + 3 \cdot 1^2) = -1.$$

The equation of the tangent line at $x = 1$ is

$$y = 1 + -1(x - 1).$$

5) To find y' we need to use the Chain Rule along with the Power Rule. Rewrite $y = \sqrt{x^2 + 2x + 1}$ as $y = (x^2 + 2x + 1)^{1/2}$. Using the Chain Rule, the Power Rule, and the fact that $(x^2 + 2x + 1)' = 2x + 2$:

$$y' = \frac{1}{2}(x^2 + 2x + 1)^{-1/2}(x^2 + 2x + 1)' = \frac{1}{2}(x^2 + 2x + 1)^{-1/2}(2x + 2).$$

To finish the problem we need to compute $y'(5)$. First rewrite $y' = \frac{2x+2}{2\sqrt{x^2+2x+1}}$.

$$y'(5) = \frac{2(5) + 2}{2\sqrt{5^2 + 2 \cdot 5 + 1}} = \frac{12}{2\sqrt{36}} = \frac{12}{12} = 1.$$

Remark: Some may have noticed that x^2+2x+1 factors as $(x+1)^2$ and so $\sqrt{x^2 + 2x + 1}$ can be rewritten as $\sqrt{(x + 1)^2}$. It is tempting to simplify further but we would need to know the sign of $x+1$ in order to "cancel" the square and the square root. The calculation above avoids this algebraic technicality.

6) We must find the slope of the secant line through two points. The first point has t-coordinate 2. Using the graph of $w(t)$, we approximate $w(2) \approx 880$ pounds. The second point will be the point whose w-coordinate is the largest. Using the graph of $w(t)$, the largest w-coordinate appears to be about 1180 pounds and this occurs when $t = 8$ (see Figure 1 below). We now compute the slope of the line through the points $(2, 880)$ and $(8, 1180)$.

$$\frac{880 - 1180}{2 - 8} = \frac{-300}{-6} = 50$$

The approximate average rate of change in weight with respect to time from age two to the time when she weighed the most is 50 pounds per year.

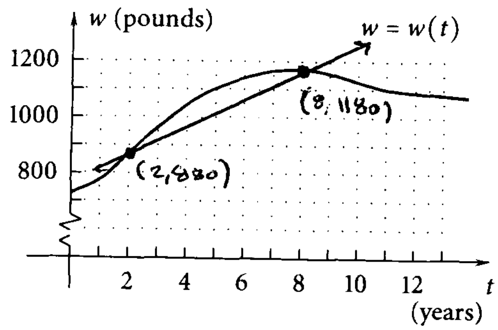

FIGURE 1

7) We are given the graph of $V = V(h)$ as a function of h but we are asked to compute a derivative of V with respect to time. To compute this derivative we need to use the chain rule which, in this case, states:

$$\frac{dV}{dt} = V'(h(t))h'(t).$$

We are given that $h(t_0) = 30$ and $h'(t_0) = 3$ where t_0 denotes the time at which the height of water is rising at 3 feet per day and the water is 30 feet deep. Thus

$$\left.\frac{dV}{dt}\right|_{t=t_0} = V'(h(t_0))h'(t_0) = V'(30) \cdot 3.$$

To finish the computation we need to find $V'(30)$. At this point we use the graph of V. $V'(30)$ is the slope of the tangent line to the graph of V at $h = 30$. To estimate this value we draw the tangent line at this point and then compute its slope by choosing two points on the tangent line (see Figure 2 below). Two points on the tangent line are $(40, 900)$ and $(30, 500)$. Thus

$$V'(30) \approx \frac{500 - 900}{30 - 40} = 40.$$

We conclude the approximate rate of change of volume with respect to time when the water is 30 feet deep if it is rising at a rate of 3 feet per day is $40 \cdot 3 = 120$ thousand gallons per day.

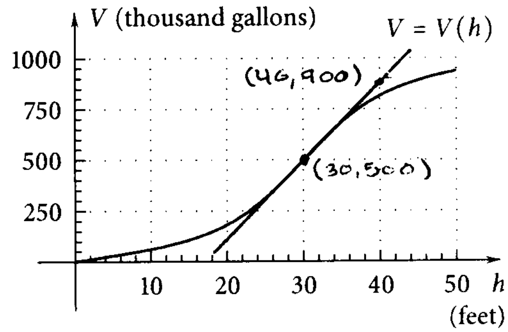
V (thousand gallons)
V = V(h)
1000
750
500
250
(46,900)
(30,500)
10
20
30
40
50
h
(feet)

FIGURE 2