Exam 2 Solutions

1) (a) Using the formula $\frac{d}{du}\ln u = \frac{1}{u}$ and the Chain Rule,

$$y'(x) = \frac{1}{x^3+x^2} \cdot \frac{d}{dx}(x^3+x^2) = \frac{3x^2+2x}{x^3+x^2}.$$

(b) Using the formula $\frac{d}{du}\cos u = -\sin u$ and the Chain Rule,

$$\frac{d}{dx}(\cos^2 x) = 2\cos x \cdot \frac{d}{dx}(\cos x) = 2\cos x(-\sin x) = -2\cos x \sin x.$$

(c) Since $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$,

$$f'(2) = \frac{1}{1+2^2} = \frac{1}{5}.$$

2) The equation of the tangent line at $x=4$ is $y = y(4) + y'(4)(x-4)$.

$$y(4) = \sqrt{4^2+9} = \sqrt{25} = 5.$$

To compute y' we use the Chain Rule and the Power Rule.

$$y' = \frac{1}{2}(x^2+9)^{-1/2}\frac{d}{dx}(x^2+9) = \frac{1}{2}(x^2+9)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+9}}.$$

Therefore $y'(4) = \frac{4}{\sqrt{4^2+9}} = \frac{4}{\sqrt{25}} = \frac{4}{5}$ and the equation of the tangent line to y at $x=4$ is

$$y = 5 + \frac{4}{5}(x-4).$$

3) We are given that we are driving 60 miles per hour and the car is using gas at $\frac{1}{20}$ gallons per mile. Let s be the distance traveled and let G be the amount of gas used on the trip. Then the rate at which we are using gasoline is

$$\frac{dG}{dt} = \frac{dG}{ds} \cdot \frac{ds}{dt} = \frac{1 \text{ gallon}}{20 \text{ miles}} \cdot \frac{60 \text{ miles}}{1 \text{ hour}} = 3 \text{ gallons per hour.}$$

4) The volume of the box is $\frac{1}{2}$ cubic foot so that $\frac{1}{2} = w^2 h$. Solving this equation for h gives $\frac{1}{2w^2} = h$. The total amount of material needed is the total area of the bottom and sides of the box. If A represents the total area then using the equation $\frac{1}{2w^2} = h$ we can write

$$A = w^2 + 4wh = w^2 + 4w\left(\frac{1}{2w^2}\right) = w^2 + \frac{2}{w}.$$

Now we optimize $A = A(w)$ for $w > 0$.

$$A'(w) = 2w - 2w^{-2} = 2w - \frac{2}{w^2}.$$

To find the critical points of A we solve $A'(w) = 0$.

$$\begin{aligned} 2w - \frac{2}{w^2} &= 0 \\ 2w &= \frac{2}{w^2} \\ w &= \frac{1}{w^2} \\ w^3 &= 1 \\ w &= 1. \end{aligned}$$

Thus $w = 1$ is the only critical point of A. It remains to verify that $A(1)$ is indeed the absolute minimum of A. We will show that A is decreasing on $(0, 1)$ and increasing on $(1, \infty)$ which implies $A(1)$ is the absolute maximum of A. Since $w = 1$ is the only critical point of A, we partition the interval $(0, \infty)$ into intervals $(0, 1)$ and $(1, \infty)$. Choose test point $\frac{1}{2}$ from the interval $(0, 1)$ and test point 2 from the interval $(1, \infty)$. It is easy to check $A'(\frac{1}{2}) < 0$ and $A'(2) > 0$ and so A is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

The desired dimensions are $w = 1$ foot and $h = \frac{1}{2(1)^2} = \frac{1}{2}$ foot.

5) (a) Using the fact that similar triangles have equal base to height ratios,

$$\frac{x + y}{18} = \frac{y}{6}. \tag{1}$$

(b) We are asked to find $\frac{dy}{dt}$ if $\frac{dx}{dt} = 2$. Simplifying (1) gives

$$\begin{aligned} 6x + 6y &= 18y \\ 6x &= 12y \\ x &= 2y. \end{aligned}$$

Differentiate both sides of this equation with respect to t to get $\frac{dx}{dt} = 2\frac{dy}{dt}$. Upon substituting $\frac{dx}{dt} = 2$ we get

$$2 = 2\frac{dy}{dt}.$$

The shadow is growing at 1 mile per hour when the man is walking 2 miles per hour.

6) Differentiate both sides of $V = w^3$ with respect to time t to obtain

$$\frac{dV}{dt} = 3w^2\frac{dw}{dt}. \tag{2}$$

We are asked to find $\frac{dw}{dt}$ when $w = 2$ and $\frac{dV}{dt} = 36$. Substituting these values in to (2) gives

$$36 = 3(2)^2\frac{dw}{dt}.$$

Therefore $\frac{dw}{dt} = \frac{36}{3\cdot(2)^2} = 3$ and the width is increasing at 3 inches per minute when the width is 2 inches and the volume is increasing at 36 cubic inches per minute.

7) (a) Since f is an odd degree polynomial with positive leading coefficient,

$$\lim_{x\to-\infty} f(x) = -\infty \text{ and } \lim_{x\to\infty} f(x) = \infty.$$

(b) In order to determine the intervals of increasing and decreasing we need to determine the sign of f'.

$$f'(x) = 3x^2 - 6x.$$

Now find the critical points of f by solving $f'(x) = 0$.

$$0 = 3x^2 - 6x = 3x(x-2).$$

$f'(x) = 0$ when $x = 0$ or $x = 2$. Partition the real line into the intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$. Choose test point -1 from $(-\infty, 0)$, test point 1 from $(0, 2)$, and test point 3 from $(2, \infty)$.

$$f'(-1) = 3(-1)^2 - 6(-1) > 0 \qquad f'(1) = 3(1)^2 - 6(1) < 0 \qquad f'(3) = 3(3)^2 - 6(3) > 0.$$

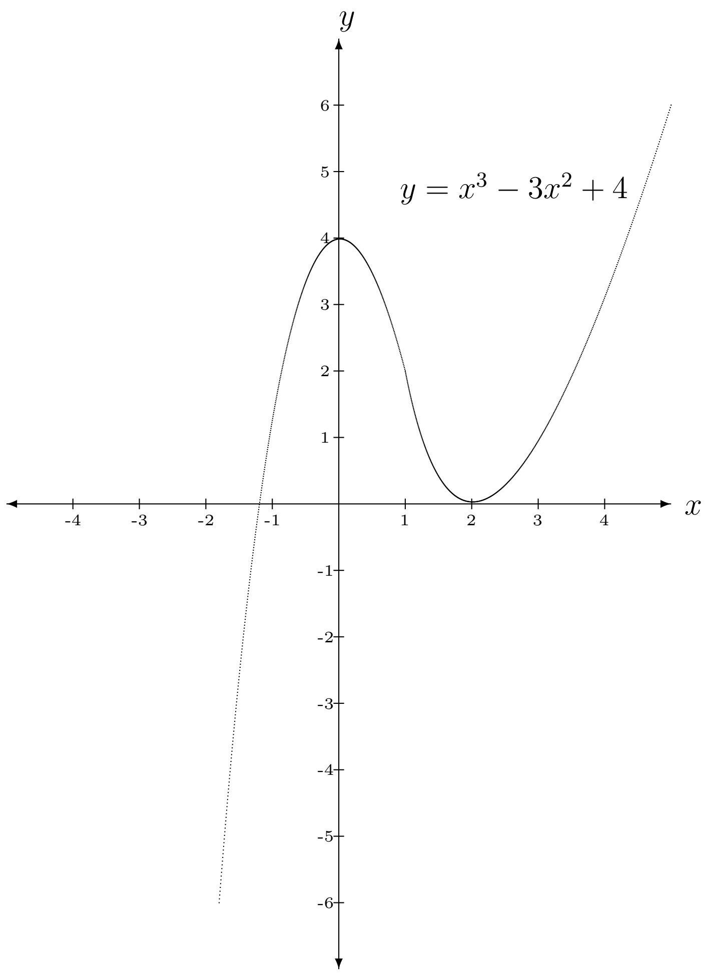
Therefore f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 2)$.

(c) In order to determine the inflection points and concavity we need to determine the sign of f''.

$$f''(x) = 6x - 6.$$

Solving $f''(x) = 0$ gives $x = 1$ and it is clear that $f''(x) > 0$ if $x > 1$ and $f''(x) < 0$ if $x < -1$. Since there is a change in sign of f'' at $x = 1$, $x = 1$ is an inflection point. f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.

(d)



8) (a) Since $y = \sin x$ is continuous, $\lim_{x\to\pi} \sin x = \sin \pi = 0$ and $\lim_{x\to\pi} \sin x = \sin(2\pi) = 0$. The limit is an indeterminate form $\frac{0}{0}$. Applying l'Hospital's Rule,

$$\begin{aligned}
\lim_{x\to\pi}\left(\frac{\sin x}{\sin(2x)}\right) &= \lim_{x\to\pi}\left(\frac{\frac{d}{dx}\sin x}{\frac{d}{dx}\sin(2x)}\right) = \lim_{x\to\pi}\left(\frac{\cos x}{\cos(2x)\frac{d}{dx}(2x)}\right)\\
&= \lim_{x\to\pi}\left(\frac{\cos x}{\cos(2x)\cdot 2}\right) = \frac{\cos\pi}{2\cos(2\pi)}\\
&= \frac{-1}{2\cdot 1} = -\frac{1}{2}.
\end{aligned}$$

(b) $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 + 1 = \infty$ therefore the limit is an indeterminate form $\frac{\infty}{\infty}$. As in (a) we apply l'Hospital's Rule.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{\ln x}{x^2+1}\right) &= \lim_{x\to\infty}\left(\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^2}\right)\\
&= \lim_{x\to\infty}\frac{1/x}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0.
\end{aligned}$$