Math 20A. Lecture 10.

Theorem 1 (L'Hopital's Rule) *Suppose that $f'(x)$ and $g'(x)$ exist with $g'(x) \neq 0$ for $x \neq x_0$ in an open interval containing x_0, that either $f(x)$ and $g(x)$ both tend to 0 or both tend to $\pm\infty$ as $x \to x_0$, and that $f'(x)/g'(x) \to L$ as $x \to x_0$. Then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L.$$

In this theorem, L may be a number or $\pm\infty$ and the condition $x \to x_0$ may be replaced by $x \to \infty$ or $x \to -\infty$ with appropriate changes in the intervals used in the definition.

(This result is called L'Hopital's rule because it was first published in *Analyse des Infiniments Petits pour l'Intelligence des Lignes Courbes* by the Marquis G. F. A. de L'Hopital in 1696. This book was one of the first textbooks on differential calculus. The result, however, was not proved by de l'Hopital but by another mathematician, Jean Bernoulli (brother of Jacob, father of Daniel), who received a regular salary from the marquis in exchange for the rights to Bernoulli's mathematical discoveries.)

The quotient $\dfrac{f(x)}{g(x)}$ in the theorem is called AN INDETERMINATE FORM OF TYPE $\dfrac{0}{0}$ if $f(x)$ and $g(x)$ tend to zero and an INDETERMINATE FORM OF TYPE $\dfrac{\infty}{\infty}$ if $f(x)$ and $g(x)$ tend to $\pm\infty$.

Example 1 Use l'Hopital's Rule to find the limit $\displaystyle\lim_{x\to 1} \frac{\sin(2\pi x)}{x^2-1}$.

Answer: $y = \sin(2\pi x) \to 0$ and $y = x^2 - 1 \to 0$ as $x \to 1$. • The quotient $\dfrac{\sin(2\pi x)}{x^2-1}$ is an indeterminate form of type $\dfrac{0}{0}$ as $x \to 1$. • $\displaystyle\lim_{x\to 1} \frac{\sin(2\pi x)}{x^2-1} = \lim_{x\to 1} \frac{\frac{d}{dx}[\sin(2\pi x)]}{\frac{d}{dx}(x^2-1)} = \lim_{x\to 1} \frac{\cos(2\pi x)\frac{d}{dx}(2\pi x)}{2x} = \lim_{x\to 1} \frac{2\pi\cos(2\pi x)}{2x} = \frac{2\pi}{2} = \pi$

L'Hopital's Rule has a special geometric interpretation in cases where the functions f and g have contunuous derivatives at the limiting value x_0 of x and $g'(x_0)$ is not zero. Then the tangent lines $y = f(x_0) + f'(x_0)(x - x_0)$ and $y = g(x_0)(x - x_0)$ approximate the graphs closely near x_0 and $f(x_0)$ and $g(x_0)$ are zero, so that. for x near x_0,

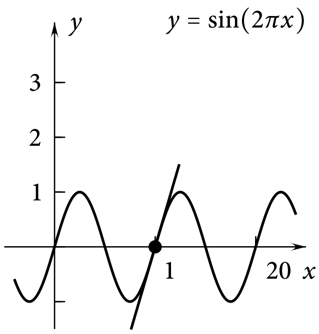
$$f(x) \approx f'(x_0)(x - x_0)$$
$$g(x) \approx g'(x_0)(x - x_0).$$

These approximations are accurate enough so that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x_0)(x-x_0)}{g'(x_0)(x-x_0)} = \frac{f'(x_0)}{g'(x_0)}.$$

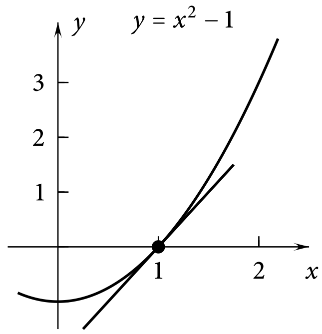
Thus, the limit of the quotient of the functions as $x \to x_0$ is equal to the ratio of the slopes of their tangent lines at x_0.

This is illustrated in Figures 1 through 3 below for the functions of Example 1. The tangent line to $y = \sin(2\pi x)$ at $x = 1$ in Figure 1 has slope 2π, the tangent line to $y = x^2 - 1$ at $x = 1$ in Figure 2 has slope 2, and the limit as $x \to 1$ of the quotient $\dfrac{\sin(2\pi x)}{x^2-1}$ in Figure 3 as $x \to 0$ is $\dfrac{2\pi}{2} = \pi$.



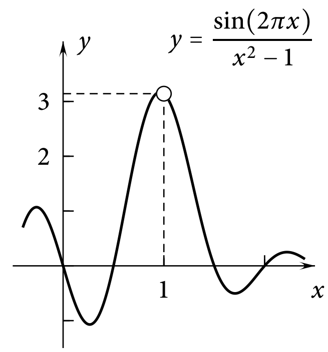
Tangent line of slope 2π

FIGURE 1



Tangent line of slope 2

FIGURE 2



$y \to \frac{2\pi}{2} = \pi$ as $x \to 1$

FIGURE 3

In the next example L'Hopital's Rule is applied twice.

Example 2 Find $\lim_{x\to 1} \frac{x^2 - 2x + 1}{x - 1 - \ln x}$.

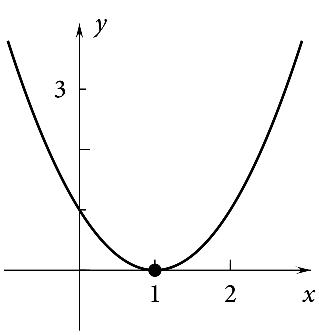
Answer: $y = \frac{x^2 - 2x + 1}{x - 1 - \ln x}$ is an indeterminate form of type $\frac{0}{0}$ as $x \to 1$ since $y = x^2 - 2x + 1$ and $y = x - 1 - \ln x$ tend to zero as $x \to 1$. •

$$\lim_{x\to 1} \frac{x^2 - 2x + 1}{x - 1 - \ln x} = \lim_{x\to 1} \frac{\frac{d}{dx}(x^2 - 2x + 1)}{\frac{d}{dx}(x - 1 - \ln x)} = \lim_{x\to 1} \frac{2x - 2}{1 - x^{-1}}$$ provided the limit on the right exists. •

$\frac{2x - 2}{1 - x^{-1}}$ is also indeterminant of type $\frac{0}{0}$ as $x \to 1$ because $2x - 2 \to 0$ and $1 - x^{-1} \to 0$ as $x \to 1$. •

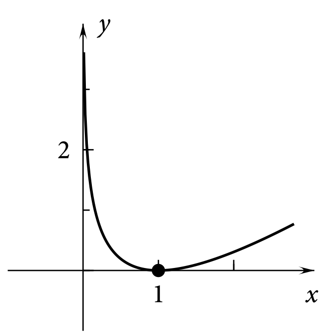
$$\lim_{x\to 1} \frac{x^2 - 2x + 1}{x - 1 - \ln x} = \lim_{x\to 1} \frac{2x - 2}{1 - x^{-1}} = \lim_{x\to 1} \frac{\frac{d}{dx}(2x - 2)}{\frac{d}{dx}(1 - x^{-1})} = \lim_{x\to 1} \frac{2}{x^{-2}} = \frac{2}{1^{-2}} = 2$$

Example 2 is illustrated in Figures 4 through 6, which show the graphs of the functions $y = x^2 - 2x + 1$ and $y = x - 1 - \ln x$ and their quotient. Notice that in this case we cannot express the limit of the quotient of the functions as the quotient of the slopes of the tangent lines to their graphs because the tangent lines (the x-axis) are horizontal and their slopes are zero.



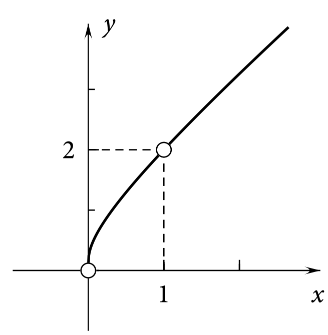
$y = x^2 - 2x + 1$

FIGURE 4



$y = x - 1 - \ln x$

FIGURE 5



$y = \frac{x^2 - 2x + 1}{x - 1 - \ln x}$

FIGURE 6

In the next example, L'Hopital's Rule does not apply and cannot be used.

Example 3 What is $\lim_{x\to 0} \frac{\sin(3x)}{1+\sin(4x)}$?

Answer: The numerator $y = \sin(3x)$ tends to 0 as $x \to 0$, but the denominator $y = 1+\sin(4x)$ tends to the nonzero number 1. • The quotient is not indeterminate. • $\lim_{x\to 0} \frac{\sin(3x)}{1+\sin(4x)} = \frac{\lim_{x\to 0}\sin(3x)}{\lim_{x\to 0}[1+\sin(4x)]} = \frac{0}{1} = 0$

Mistakenly applying l'Hopital's Rule to Example 3 would lead to the incorrect result,

$$\lim_{x\to 0} \frac{\sin(3x)}{1+\sin(4x)} = \lim_{x\to 0} \frac{\frac{d}{dx}[\sin(3x)]}{\frac{d}{dx}[1+\sin(4x)]} = \lim_{x\to 0} \frac{\cos(3x)\frac{d}{dx}(3x)}{\sin(4x)\frac{d}{dx}4x)} = \lim_{x\to 0} \frac{3\cos(3x)}{4\cos(4x)} = \tfrac{3}{4}.$$

Example 4 Find $\lim_{x\to\infty} \frac{e^{x/3}}{x}$.

Answer: $y = \frac{e^{x/3}}{x}$ is an indeterminate of type $\frac{\infty}{\infty}$ as $x \to \infty$ because $e^{x/2}$ and x both tend to ∞. •

$$\lim_{x\to\infty} \frac{e^{x/2}}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^{x/2})}{\frac{d}{dx}(x)} = \lim_{x\to\infty} \frac{e^{x/2}\frac{d}{dx}(\frac{1}{2}x)}{1} = \lim_{x\to\infty} \tfrac{1}{2}e^{x/2} = \infty$$

This example ilustrates the general principle that exponential functions $y = e^{ax}$ for $a > 0$ tend to ∞ faster than all polynomials as $x \to \infty$.

Example 5 Sketch the graph of $y = \frac{e^{x/2}}{x}$ by studying the function and its first derivative.

Answer: Study the function. • $y = \frac{e^{x/2}}{x}$ is defined and continuous for $x \neq 0$, is positive for $x > 0$ and is negative for $x < 0$. • It tends to ∞ as $x \to \infty$ by Example 4. • It tends to 0 as $x \to -\infty$ because e^x and $1/x$ both tend to zero. • It tends to $-\infty$ as $x \to 0^-$ because $e^{x/2} \to 1$ and $1/x \to -\infty$. • It tends to ∞ as $x \to 0^+$ because $e^{x/2} \to 1$ and $1/x \to \infty$. • Figure 7 •

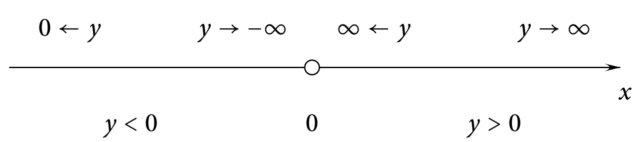


FIGURE 7

Study the derivative:

$$y' = \frac{d}{dx}\left(\frac{e^{x/2}}{x}\right) = \frac{x\frac{d}{dx}(e^{x/2}) - e^{x/2}\frac{d}{dx}(x)}{x^2} = \frac{xe^{x/2}\frac{d}{dx}\left(\frac{1}{2}x\right) - e^{x/2}}{x^2} = \left(\frac{\frac{1}{2}x - 1}{x^2}\right)e^{2/x} = \left(\frac{x-2}{2x^2}\right)e^{2/x} \quad \bullet$$

The derivative is not defined at $x = 0$, is zero at $x = 2$, is negative for $x < 0$ and for $0 < x < 2$, and is positive for $x > 0$. • The function is decreasing on $(-\infty, 0)$ and on $(0, 2]$, is increasing on $[2, \infty)$, and has a local minimum at $x = 2$. • Figure 8

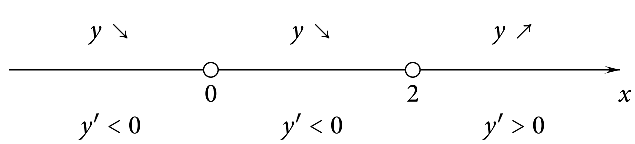


FIGURE 8

Draw the graph. • One approach: Use a calculator to find the values $y(-1) = -e^{-1/2} \doteq -0.61$, $y(1) = e^{1/2} \doteq 1.65$, $y(2) = \frac{1}{2}e \doteq 1.36$, and $y(6) = \frac{1}{6}e^3 \doteq 3.35$ and plot these points. • Figure 9

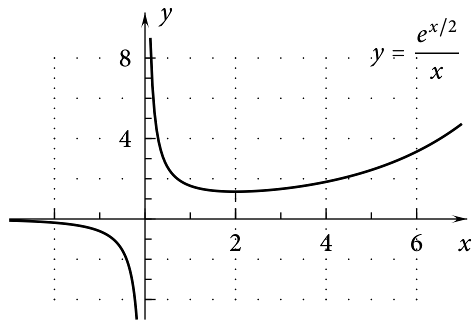


FIGURE 9

***Example** 6* Show that $\dfrac{\ln x}{\sqrt[n]{x}} \to 0$ as $x \to \infty$ for every positive integer n.

Answer: $y = \dfrac{\ln x}{\sqrt[n]{x}}$ is an indeterminate form of type ∞/∞ as $x \to \infty$ because $\ln x \to \infty$ and $\sqrt[n]{x} \to \infty$ as $x \to \infty$. •

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/n}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{1/n})} = \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{n}x^{(1/n)-1}} = \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{n}x^{(1/n)}x^{-1}} = \lim_{x\to\infty} \frac{n}{x^{1/n}} = 0$$

Indeterminate forms of types $0 \cdot \infty$

The product $y = f(x)g(x)$ of a function f that tends to 0 and a function g that tends to ∞ is called an INDETERMINATE FORM OF TYPE $0 \cdot \infty$. The limits of many such expressions can be found by rewriting them as indeterminate forms of type $0/0$ or ∞/∞.

***Example** 7* Find $\lim_{x\to 0^+} [\sqrt{x}\,(\ln x)]$.

Answer: $y = \sqrt{x}\,(\ln x)$ is indeterminate of type $0 \cdot \infty$ as $x \to 0^+$ because $y = \sqrt{x} \to 0$ and $y = \ln x \to -\infty$. •
Rewrite $\sqrt{x}\,(\ln x)$ as $\dfrac{\ln x}{x^{-1/2}}$, which is an indeterminate form of type ∞/∞ as $x \to 0^+$. •

$$\lim_{x\to 0^+} [\sqrt{x}\,(\ln x)] = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-1/2})} = \lim_{x\to 0^+} \frac{x^{-1}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} (-2x^{1/2}) = 0$$

We could attempt to find the limit in Example 7 by writing the given expression in the form $\dfrac{\sqrt{x}}{(\ln x)^{-1}}$, which is of type $0/0$ as $x \to 0^+$. Then, however, it does not help to apply l'Hopital's Rule because the quotient of the derivatives is $\dfrac{\frac{d}{dx}(x^{1/2})}{\frac{d}{dx}[(\ln x)^{-1})]} = \dfrac{\frac{1}{2}x^{-1/2}}{-(\ln x)^{-2}x^{-1}} = \dfrac{-\sqrt{x}}{2(\ln x)^{-2}} = -\frac{1}{2}\sqrt{x}(\ln x)^2$, and we cannot tell what its limit is as $x \to 0^+$. Nor would it help to apply l'Hopital's Rule to the new quotient.

Indeterminate forms of types $0^0, 1^\infty$ and ∞^0

The expression $y = f(x)^{g(x)}$ is INDETERMINATE OF TYPE 0^0 if $f(x) \to 0$ and $g(x) \to 0$; it is OF TYPE 1^∞ if $f(x) \to 1$ and $g(x) \to \pm\infty$; and it is OF TYPE ∞^0 if $f(x) \to \infty$ and $g(x) \to 0$. The limits of such expressions can often by found by applying l'Hopital's Rule to their logarithms.

Example *8* Find the limit of $y = x^{-x}$ as $x \to 0^+$.

Answer: $y = x^{-x}$ is of type 0^0 as $x \to 0^+$. • $\ln(x^{-x}) = -x\ln x$ is an indeterminate of type $0 \cdot \infty$. • $\lim_{x\to 0^+} \ln(x^{-x}) = \lim_{x\to 0^+} (-x\ln x) = \lim_{x\to 0^+} \dfrac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \dfrac{\frac{d}{dx}(-\ln x)}{\frac{d}{dx}(x^{-1})} = \lim_{x\to 0^+} \dfrac{-x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} x = 0$ • Because its logarithm tends to 0, the function $y = x^{-x} = e^{\ln(x^{-x})}$ tends to $e^0 = 1$ as $x \to 0^+$/ • $\lim_{x\to 0^+} x^{-x} = 1$ • The curve $y = x^{-x}$ is shown in Figure 10.

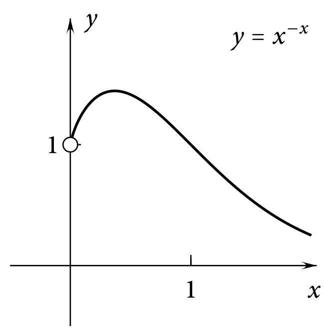


FIGURE 10

Piecewise constant rates of change

Example *9* At 10:00 AM one morning a truck driver is 50 miles east of a town He drives 75 miles per hour toward the east for two hours to make a delivery. Next, he drives west at 50 miles per hour for two hours to make another delivery and then drives east at 50 miles per hour for two more hours. According to this mathematical model, his velocity toward the east is the "step" function of Figure 11 with $t = 0$ at 10 AM. **(a)** How far is he from the first town at $t = 6$? **(b)** How is the answer to part (a) related to the areas of the rectangles in Figure 12?

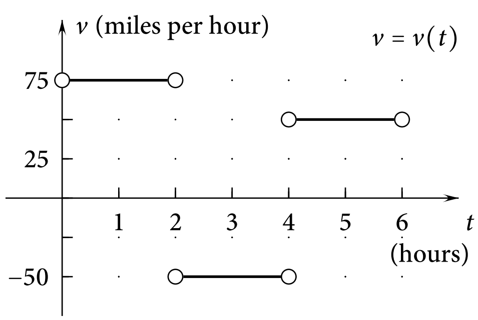


FIGURE 11

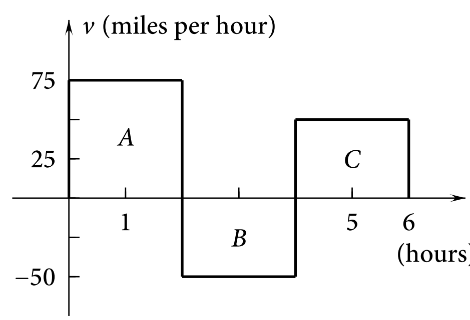


FIGURE 12

Answer: **(a)** At $t = 6$, he is [50 miles] + [75 miles per hour][2 hours] − [50 miles per hour][2 hours] + [50 miles per hour][2 hours] = 50 + 150 − 100 + 100 = 200 miles east of the town.
(b) His location at $t = 6$ equals his location at $t = 0$ plus the the sum of the areas of rectangles A and C in Figure 12, minus the area of rectangle B.

Partitions and step functions

A PARTITION of a finite closed interval $[a, b]$ is a finite number of points $x_0, x_1, x_2, x_3, \ldots, x_N$ such that

$$a = x_0 < x_1 < x_2 < x_3 \cdots < x_N = b.$$

These points divide $[a, b]$ into N subintervals. Figure 13 shows, for example, the four subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, and $[x_3, x_4]$ that are defined by a partition, $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$.

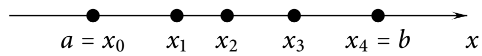


FIGURE 13

A function is a STEP FUNCTION on an interval $[a, b]$ if it is constant on the interiors (x_0, x_1), (x_1, x_2), $(x_2, x_3), \ldots, (x_{N-1}, x_N)$ of the subintervals in a partition of the interval. The function might or might not be defined at endpoints of the subintervals.

Theorem 2 *Suppose that a function $y = F(x)$ is continuous on a finite closed interval $[a, b]$ and that its derivative $r = F'(x)$ is a step function on $[a, b]$. Then the region between the graph $r = F'(x)$ and the x-axis for $a \le x \le b$ consists of a finite number of rectangles, and the change in the function's value from $x = a$ to $x = b$ is given by*

$$F(b) - F(a) = \begin{bmatrix} \text{The area of} \\ \text{all rectangles} \\ \text{above the } x\text{-axis} \end{bmatrix} - \begin{bmatrix} \text{The area of} \\ \text{all rectangles} \\ \text{below the } x\text{-axis} \end{bmatrix}.$$

Example 10 A water tank contains 300 gallons of water at time $t = 0$ (minutes) and the rate of flow $r = r(t)$ (gallons per minute) into the tank for $0 \le t \le 70$ is the step function in Figure 14. How much water is in the tank at $t = 70$?

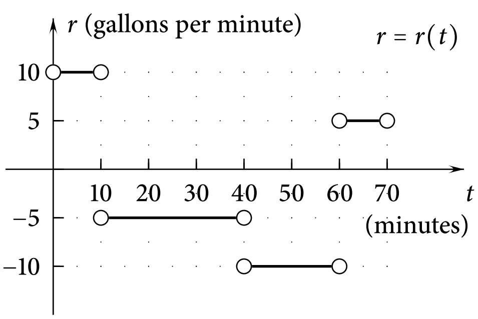


FIGURE 14

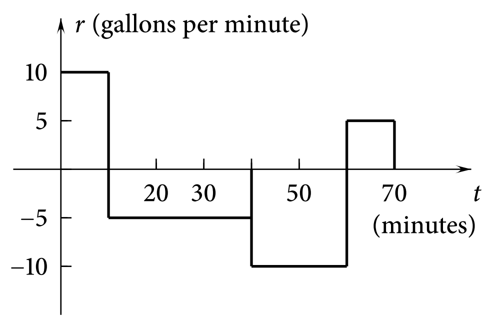


FIGURE 15

Answer: We assume that the volume $V(t)$ of water in the tank at time t is continuous on $[0, 70]$. •

$V(70) - V(0) = \left[10\ \dfrac{\text{gallons}}{\text{minute}}\right][10 \text{ minutes}] + \left[5\ \dfrac{\text{gallons}}{\text{minute}}\right][10 \text{ minutes}] - \left[5\ \dfrac{\text{gallons}}{\text{minute}}\right][30 \text{ minutes}]$

$- \left[10\ \dfrac{\text{gallons}}{\text{minute}}\right][20 \text{ minutes}] = 100 + 50 - 150 - 200 \text{ gallons} = -200 \text{ gallons}.$ •

$V(70) = V(0) - 200 = 300 - 200 = 100$ • There are 100 gallons of water in the tank at $t = 70$

The volume of water in the tank at $t = 70$ in Example 10 is equal to the volume at $t = 0$ plus the area of the two rectangles above the t-axis in Figure 15, minus the area of the two rectangles below the t-axis.

Example 11 What is $F(1)$ if $y = F(x)$ is continuous on $[0,1]$, $F(0) = 10$, and $F'(x)$ equals 10 for $0 < x < \frac{1}{3}$, equals 20 for $\frac{1}{3} < x < \frac{2}{3}$, and equals 30 for $\frac{1}{3} < x < 1$?

Answer: $F(1) = F(0) + 10(\frac{1}{3}) + 20(\frac{1}{3}) + 30(\frac{1}{3}) = 10 + \frac{60}{3} = 30$

Example 12 A few plants, including jack in the pulpit, skunk cabbage, and sacred lotus, can create heat from oxygen and other nutrients. The step function in Figure 16 is a model of the rate of consumption of oxygen by a sacred lotus plant from noon one day to midnight 36 hours later. The flower kept its temperature between 30°C and 37°C as the air temperature varied between 10°C and 35°C by consuming more oxygen in the colder periods. (Data adapted from *Scientific American*) How much oxygen did the flower consume during the 36 hours? (t is measured in minutes.)

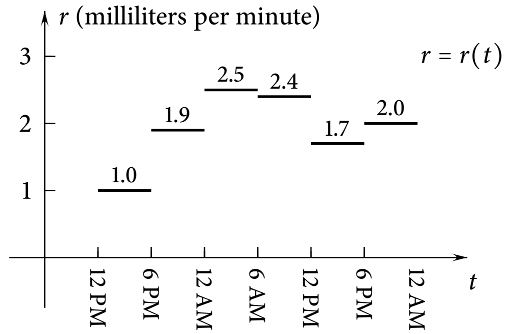


FIGURE 16

Answer: [Total oxygen consumed]=$1(360) + 1.9(360) + 2.5(360) + 2.4(360) + 1.7(360) + 2.0(360) = 4140$ milliliters

Example 13 Figures 17 and 18 show the graphs of a piecewise linear function $y = f(x)$ and its derivative $r = f'(x)$. According to the conclusion of Theorem 2, $f(3) - f(0)$ is equal to the combined area $2(1) + 1(2) = 4$ of rectangles A and B in Figure 18. Instead, $f(3) - f(0) = 2 - 0 = 2$. Explain.

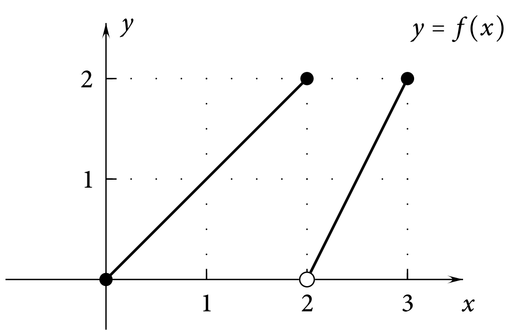


FIGURE 17

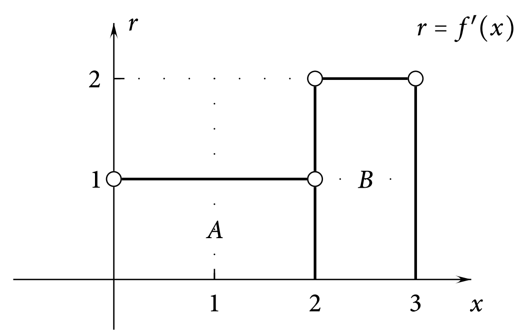


FIGURE 18

Answer: Theorem 2 does not apply because f is not continuous on $[0.3]$.

Approximating continuous rates of change with step functions

Suppose that the continuous function $v = v(t)$ of Figure 19 is the velocity of a car that is at $s = s(t)$ on an s-axis at time t, so that $v(t) = s'(t)$ for $a < t < b$ and that $s = s(t)$ is continuous on $[a, b]$. Notice that the region between the graph and the t axis for $a \le t \le b$ in Figure 19 is in two parts. The region labeled A for the time period $a \le t < c$, when the car's velocity is positive, is above the t-axis. The region labeled B for $c < t \le b$, when the car's velocity is negative, is below the t-axis.

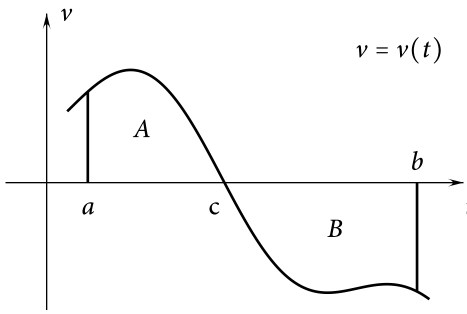


FIGURE 19

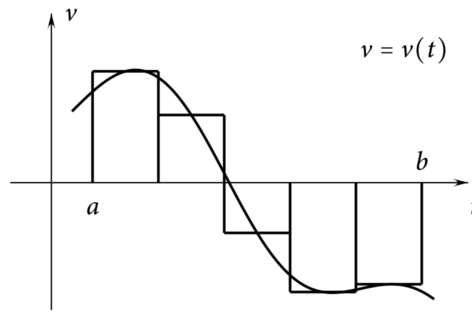


FIGURE 20

We cannot apply Theorem 2 in this case because the velocity is not a step function. Instead we approximate $v = v(t)$ with a step function by approximating regions A and B by rectangles, as in Figure 20, where the sides of the rectangles are determined by a partition of $[a, b]$ and the tops are chosen to intersect the graph of v. If the car's velocity were given by the step function, we could apply Theorem 2 and conclude that the change in the car's position $s(b) - s(a)$ from $t = a$ to $t = b$ equals the area of the two rectangles above the t-axis, minus the area of the three rectangles below the t-axis. Instead, the step function approximates the actual velocity v, and the difference of the areas of the rectangles approximates the change in the car's position:

(1)
$$s(b) - s(a) \approx \begin{bmatrix} \text{The area of} \\ \text{all rectangles} \\ \text{above the } t\text{-axis} \end{bmatrix} - \begin{bmatrix} \text{The area of} \\ \text{all rectangles} \\ \text{below the } t\text{-axis.} \end{bmatrix}.$$

Formula (1) would give a better approximation if we used more, narrower rectangles, as in Figure 21. To obtain exact results, we let the number of rectangles tend to infinity and their widths tend to zero. We can expect that the areas of the two sets of rectangles determined by the graph of the continuous function $v = v(t)$ would approach the areas of regions A and B in Figure 19, and that the change in position with velocity given by the step function would approach the change in position of the truck with velocity given by $v(t)$. Consequently, we can expect that

(2)
$$\begin{aligned} s(b) - s(a) &= \lim \left\{ \begin{bmatrix} \text{The area of} \\ \text{all rectangles} \\ \text{above the } t\text{-axis} \end{bmatrix} - \begin{bmatrix} \text{The area of} \\ \text{the rectangles} \\ \text{below the } t\text{-axis} \end{bmatrix} \right\} \\ &= \begin{bmatrix} \text{The area of region} \\ A \text{ in Figure 20} \end{bmatrix} - \begin{bmatrix} \text{The area of region} \\ B \text{ in Figure 20} \end{bmatrix} \end{aligned}$$

where the limit is taken as the number of rectangles in the approximation tends to ∞ and their widths tend to 0.

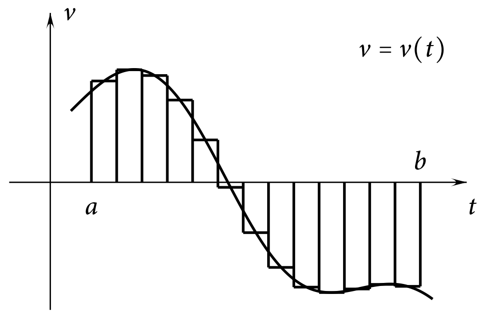


FIGURE 21

In the next lecture we will define the INTEGRAL $\int_a^b v(t)\,dt$ so that it equals the difference of areas on the right of (2), and we will see that with this definition, equation (2) is an example of Part 1 of the FUNDAMENTAL THEOREM OF CALCULUS.

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/
(The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 5.1: 1–9

Section 6.1: 1, 2