Math 20A. Lecture 11.

The DEFINITE INTEGRAL of the function $y = f(x)$ from $x = a$ to $x = b$ with $a < b$ is a number, denoted $\int_a^b f(x)\,dx$. The symbol $\int$ is called an INTEGRAL SIGN, the numbers a and b are the LIMITS OF INTEGRATION, $[a, b]$ is the INTERVAL OF INTEGRATION, and $f(x)$ is the INTEGRAND. We assume that f is defined at all but possibly a finite number of points in $[a, b]$.

As was explained at the end of the last lecture, the integral is defined so that for the function f of Figure 1, the integral from $x = a$ to $x = b$ equals the area of region A between the graph and the x-axis where $f(x)$ is positive, minus the area of region B between the graph and the x-axis where $f(x)$ is negative. To accomplish this, $y = f(x)$ is approximated by step functions and the regions by collections of rectangles, as in Figure 2. The integral is defined to be the limit, as the number of rectangles tends to ∞ and their widths tend to zero, of the area of the rectangles above the x-axis, minus the area of the rectangles below the x-axis.

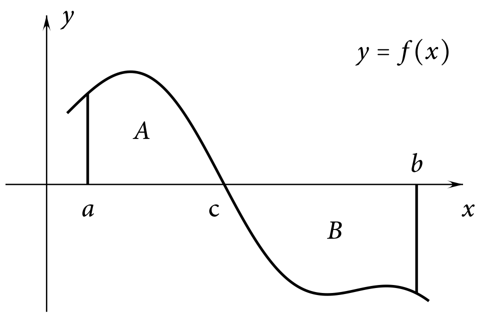


FIGURE 1

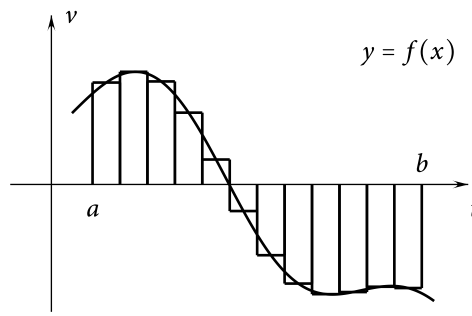


FIGURE 2

To construct the five rectangles in Figure 3, a PARTITION,

$$a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b$$

of the interval $[a, b]$ into five subintervals is selected and a point c_j is chosen in the jth subinterval for $j = 1, 2, \ldots 5$. The bottoms of the two rectangles above the x-axis are the first two subintervals on the x-axis and their tops intersect the graph at $x = c_1$ and $x = c_2$. The tops of the three rectangles below the x-axis are the last three subintervals on the x-axis and their bottoms intersect the graph at $x = c_3$, $x = c_4$, and $x = c_5$.

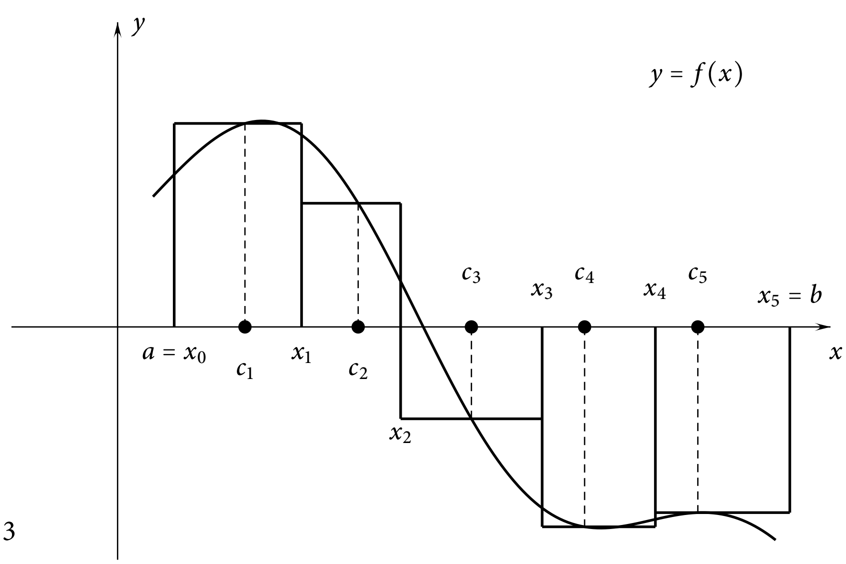
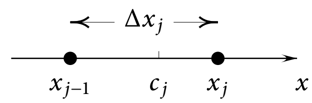


FIGURE 3

The jth subinterval is $[x_{j-1}, x_j]$ (Figure 4). We let Δx_j denote its width:

$$[\text{The width of the } j\text{th subinterval}] = \Delta x_j = x_j - x_{j-1}.$$



The jth subinterval
FIGURE 4

The heights of the two rectangles above the x-axis in Figure 3 are $f(c_1)$ and $f(c_2)$, and the heights of the three rectangles below the x-axis are $-f(c_3)$, $-f(c_4)$ and $-f(c_5)$. Consequently,

(1)
$$\begin{aligned}
&\begin{bmatrix}\text{Area of the rectangles}\\ \text{above the } x\text{-axis}\end{bmatrix} - \begin{bmatrix}\text{Area of the rectangles}\\ \text{below the } x\text{-axis}\end{bmatrix}\\
&\qquad = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 - [-f(c_3)]\,\Delta x_3 - [-f(c_4)]\,\Delta x_4 - [-f(c_5)]\,\Delta x_5\\
&\qquad = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 + f(c_5)\Delta x_5\\
&\qquad = \sum_{j=1}^{5} f(c_j)\Delta x_j.
\end{aligned}$$

We used SUMMATION NOTATION in the last expression. Notice that there are no minus signs in the last two lines of (1); the areas of the last three rectangles are subtracted because the last three values of the function f are negative.

The sum on the right of (1) is called a RIEMANN SUM for the integral $\int_a^b f(x)\,dx$. The general definition folows.

Definition 1 (Riemann sums and the definite integral) **(a)** *Consider a function $y = f(x)$ defined at all or at all but a finite number of points in an interval $[a, b]$. A Riemann sum for the integral $\int_a^b f(x)\,dx$ corresponding to a partition $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ of $[a, b]$ is a sum,*

$$\sum_{j=1}^{N} f(c_j)\Delta x_j$$

where for each $j = 1, 2, 3, \ldots, N$, c_j is a point in the jth subinterval such that $f(c_j)$ is defined and $\Delta x_j = x_j - x_{j-1}$ is the width of the jth subinterval.

(b) *The definite integral of $y = f(x)$ from $x = a$ to $x = b$ is the limit of Riemann sums,*

$$\int_a^b f(x)\,dx = \lim \sum_{j=1}^{N} f(c_j)\Delta x_j$$

as the number N of subintervals in the partitions tends to infinity and their widths tend to zero, provided that the limit exists and is finite.

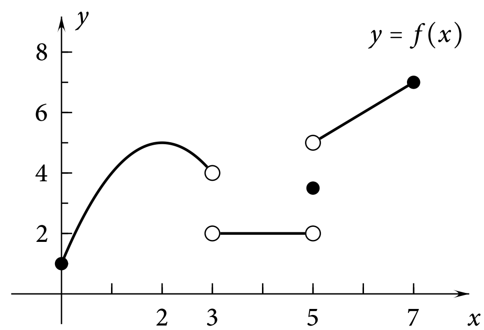
Piecewise continuous functions

We consider definite integrals of functions which are PIECEWISE CONTINUOUS, according to the following definition.

Definition 2 *The function $y = f(x)$ is* PIECEWISE CONTINUOUS *on $[a, b]$ if it is defined at all but possibly a finite number of points in $[a, b]$ and there is a partition of the interval such that f is continuous on the interior of each subinterval and has finite limits from the right at the left endpoints of the subintervals and finite limits from the left at the right endpoints.*

According to this definition, a piecewise continuous function might not be defined or might be discontinuous at a finite number of points in the interval, but it must be BOUNDED. i. e. there must be a constant M such that $|f(x)| \le M$ on $[a, b]$. The function f of Figure 5 is piecewise continuous on the interval $[0, 7]$.

FIGURE 5

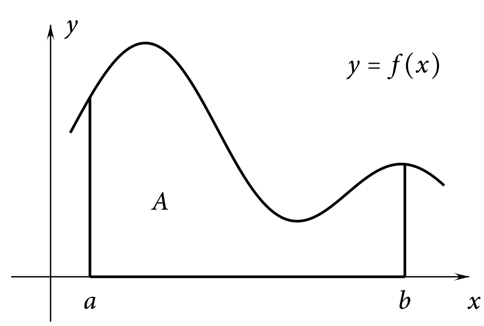


The following result is established in courses on advanced calculus.

Theorem 1 *The Riemann integral $\int_a^b f(x)\, dx$ is defined if f is piecewise continuous on the interval $[a, b]$.*

Integrals and areas

Figure 6 shows a region A between the graph of a positive, continuous function f and above the x-axis for $a \le x \le b$. In Definition 1 the integral $\int_a^b f(x)\, dx$ is defined to be the limit of areas of approximations of this region by collections of rectangles such that the approximations become increasingly accurate as the limit is taken. Accordingly, we define the area of A to be the integral:



$$[\text{Area } A] = \int_a^b f(x)\, dx$$

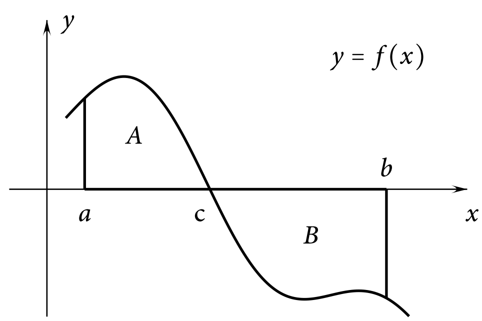
FIGURE 6

Definition 3 (Area) *If f is piecewise continuous and its values are ≥ 0 on $[a, b]$, then the area of the region between the graph $y = f(x)$ and the x-axis for $a \leq x \leq b$ equals the integral $\int_a^b f(x)\,dx$.*

This definition is consistent with other definitions of area if the region consists of rectangles, triangles, or other figures for which there are area formulas from geometry, and it defines the area in other cases.

If f has negative, or positive and negative, values in $[a, b]$, then Riemann sum approximations of $\int_a^b f(x)\,dx$ equal the area of rectangles that approximate the region between the graph and the x-axis where $f(x)$ is positive, minus the area of rectangles that approximate the region between the graph and the x-axis where $f(x)$ is negative. Accordingly, we have the following general result for a not necessarily positive function, which we anticipated earlier.

Theorem 2 (Integrals as areas) *If f is piecewise continuous on $[a, b]$, then the integral $\int_a^b f(x)\,dx$ equals the area of the regions above the x-axis and below the graph where $f(x)$ is positive, minus the area of the regions below the x-axis and above the graph where $f(x)$ is negative. (Figure 7).*



$\int_a^b f(x)\,dx$
$= [\text{Area } A] - [\text{Area } B]$
FIGURE 7

Example 1 Use the formula for the area of a triangle to evaluate $\int_{-3}^{3} (x+1)\,dx$.

Answer: $y = x + 1$ is the line in Figure 8. • $\int_{-3}^{3} (x+1)\,dx = [\text{Area } B] - [\text{Area } A] = 8 - 2 = 6$.

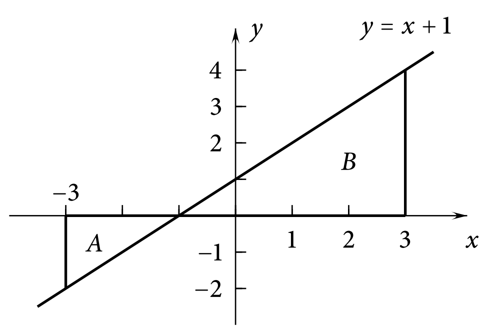


FIGURE 8

Calculating Riemann sums

A Riemann sum $\sum_{j=1}^{N} f(c_j)\Delta x_j$ for $\int_a^b f(x)\,dx$ is a RIGHT RIEMANN SUM if the points c_j are the right endpoints x_j of the subintervals of the corresponding partition $a = x_0, x_1 < x_2 < \cdots < x_N = b$. It is a LEFT RIEMANN SUM if the c_j's are the left endpoints x_{j-1}, and is a MIDPOINT RIEMANN SUM if the c_j's are the midpoints $\frac{1}{2}(x_{j-1} + x_j)$ of the subintervals.

Example 2 Give **(a)** the right Riemann sum and **(b)** the left Riemann sum for $\int_0^5 (x^3 + 5x)\,dx$ corresponding to the general partition $0 = x_0, < x_1 < x_2 < \cdots < x_N = 5$ of $[0, 5]$.

Answer: **(a)** x_j is the right endpoint of the jth subinterval. • [Right Riemann sum] = $\sum_{j=1}^{N} [(x_j)^3 - 5x_j]\Delta x_j$

(b) x_{j-1} is the left endpoint of the jth subinterval. • [Left Riemann sum] = $\sum_{j=1}^{N} [(x_{j-1})^3 - 5x_{j-1}]\Delta x_j$

If the subintervals in a partition for a Riemann sum are of equal width, as in the next example, we write Δx instead of Δx_j for that width.

Example 3 Calculate the right Riemann sum for $\int_0^1 x^2\,dx$ corresponding to the partition of $[0, 1]$ into five equal subintervals. Draw the curve $y = x^2$ with the rectangles whose areas give the Riemann sums.

Answer: **(a)** Because $[0, 1]$ has width 1, $\Delta x = \frac{1}{5} = 0.2$ and the partition is $0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$. •
The right endpoints are $0.2, 0.4, 0.6, 0.8$, and 1. • [Right Riemann sum]
$= (0.2)^2(0.2)+(0.4)^2(0.2)+(0.6)^2(0.2)+(0.8)^2(0.2)+1^2(0.2) = [(0.2)^2+(0.4)^2+(0.6)^2+(0.8)^2+1^2](0.2) = 0.44.$
• Figure 9 • (Figures 10 and 11 show the rectangles for the left and midpoint Riemann sums.)

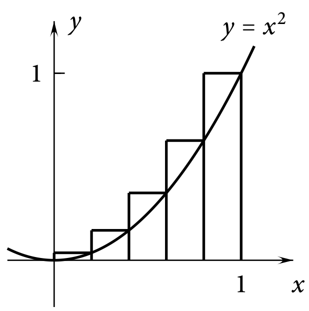


FIGURE 9

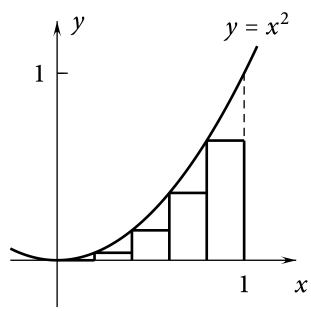


FIGURE 10

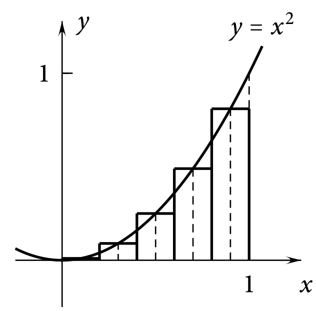


FIGURE 11

Example 4 (Not required) Use the formula, $1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{1}{6}N$, for the sum of the squares of the first N positive integers to find the value of $\int_0^1 x^2\,dx$.

Answer: One approach: Use right Riemann sums with the partition of $[0, 1]$ into N equal subintervals. •
Partition: $0 < \frac{1}{N} < \frac{2}{N} < \frac{3}{N} < \cdots < \frac{N-1}{N} < 1$ • $\Delta x = \frac{1}{N}$ • The right endpoint of the jth subinterval is $x_j = \frac{j}{N}$.
• [Right Riemann sum] = $\sum_{j=1}^{N} (x_j)^2 \Delta x$

$$= \left[\left(\frac{1}{N}\right)^2 + \left(\frac{2}{N}\right)^2 + \left(\frac{3}{N}\right)^2 + \cdots + \left(\frac{N}{N}\right)^2\right]\left(\frac{1}{N}\right) = \frac{1}{N^3}(1^2 + 2^2 + 3^2 + \cdots N^2)$$

$$= \frac{1}{N^3}\left(\tfrac{1}{3}N^3 + \tfrac{1}{2}N^2 + \tfrac{1}{6}N\right) = \tfrac{1}{3} + \tfrac{1}{2}N^{-1} + \tfrac{1}{6}N^{-2} \to \tfrac{1}{3} \text{ as } N \to \infty$$ • $\int_0^1 x^2\,dx = \frac{1}{3}$

Properties of definite integrals

Because regions of zero width have zero area, integrals with equal limits of integration are defined to be zero. Also, an integral from b to a with $a < b$ is defined to be the negative of the integral from a to b:

Definition 4 **(a)** *For any function f,*

$$\int_a^a f(x)\,dx = 0.$$

(b) *If the integral of f from a to b is defined with $a < b$, then*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Example 5 **(a)** What is the value of $\int_{10}^{10} x^2\,dx$? **(b)** In Example 4 we found that $\int_0^1 x^2\,dx = \frac{1}{3}$. What is the value of $\int_1^0 x^2\,dx$?

Answer: (a) $\int_{10}^{10} x^2\,dx = 0$ (b) $\int_1^0 x^2\,dx = -\int_0^1 x^2\,dx = -\frac{1}{3}$

Theorem 3 (Integrals over adjacent intervals) *If f is piecewise continuous on an interval containing the numbers a, b, and c, then*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

This result follows from the interpretation of integrals as areas if $f(x) \ge 0$ on the interval and $a < b < c$ (Figure 12). It can be derived for other cases using Riemann sums and Definition 4.

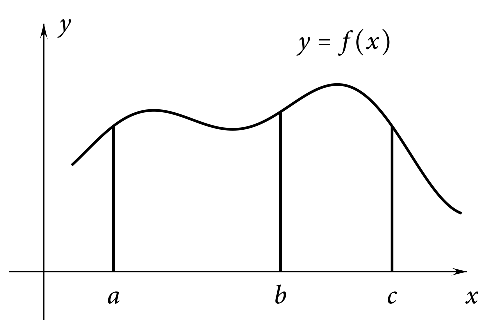


FIGURE 12

Example 6 What is the integral of g from 1 to 8 if its integral from 1 to 5 is −7 and its integral from 5 to 8 is 9?

Answer: $\int_1^8 g(x)\,dx = \int_1^5 g(x)\,dx + \int_5^8 g(x)\,dx = -7 + 9 = 2$

Theorem 4 (Integrals of linear combinations) *If f and g are piecewise continuous on an interval containng a and b, then for any constants A and B,*

$$\int_a^b [Af(x) + Bg(x)]\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx \tag{5}$$

This result holds because any Riemann sum for the integral of $y = Af(x) + Bg(x)$ equals A multiplied by a Riemann sum for the integral of f, plus B multiplied by a Riemann sum for the integral of g:

$$\sum_{j=1}^{N}[Af(c_j) + Bg(c_j)]\Delta x = A\sum_{j=1}^{N} f(c_j)\Delta x + B\sum_{j=1}^{N} g(c_j)\Delta x.$$

Formula (5) follows since the integrals are the limits of their respective Riemann sums.

Example 7 What is $\int_{-35}^{35} [2p(x) - 3q(x)]\,dx$ if $\int_{-35}^{35} p(x)\,dx = 10$ and $\int_{-35}^{35} q(x)\,dx = 20$?

Answer: $\int_{-35}^{35} [2p(x) - 3q(x)]\,dx = 2\int_{-35}^{35} p(x)\,dx - 3\int_{-35}^{35} q(x)\,dx = 2(10) - 3(20) = -40$

Unbounded functions

The Riemann integral $\int_a^b f(x)\,dx$ is not defined if f is not bounded on $[a, b]$. For example, $\int_0^1 \frac{1}{x}\,dx$ is not defined as a Riemann integral because $1/x \to \infty$ as $x \to 0^+$ and consequently $y = 1/x$ is not bounded on the interval of integration $[0, 1]$ (Figure 13). The Riemann sums for this integral cannot have a finite limit because arbitrarily large Riemann sums can be constructed by choosing c_1 in the partitions sufficiently close to 0.

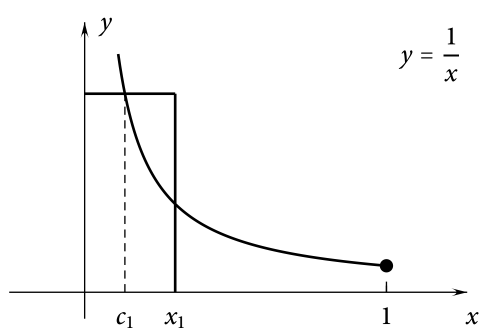


FIGURE 13

In Math 20B you will see that integrals of certain unbounded functions can be defined as IMPROPER INTEGRALS. These are limits of Riemann integrals.

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/
(The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 6.2: 1–4, 6